Stable solutions of nonlinear elliptic Cauchy problems in three dimensional domains

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Stable solutions of nonlinear elliptic Cauchy problems in three dimensional domains

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Abstract: In this article an iterative method of level-set type is proposed for solving nonlinear elliptic Cauchy problems. We provide convergence analysis including stability and convergence results. Moreover, a numerical investigation of the proposed method is presented. Our algorithms make use of a pre-conditioning strategy, which accounts for performance improvement. The whole approach is focused on three dimensional models, better suited for real life applications.

Key words: Nonlinear Cauchy problems; Elliptic operators; Level-set methods.

AMS classification: 65J20, 35J60

1 Introduction

We start by introducing the inverse problem under consideration. Let \( \Omega \subset \mathbb{R}^3 \), be an open bounded set with piecewise Lipschitz boundary \( \partial \Omega \). Further, we assume that \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \), where \( \Gamma_i \) are two open disjoint parts of \( \partial \Omega \). Given the function \( q : \mathbb{R} \to \mathbb{R}^+ \), we define the second order elliptic operator

\[
\mathcal{P}(u) := -\nabla \cdot (q(u)\nabla u).
\]

We denote by **nonlinear elliptic Cauchy problem** the following problem

\[
(CP_{nl}) \quad \mathcal{P}(u) = f, \text{ in } \Omega \quad u = g_1, \text{ at } \Gamma_1 \quad q(u) u_\nu = g_2, \text{ at } \Gamma_1,
\]

where the pair of functions \((g_1, g_2) \in H^{1/2}(\Gamma_1) \times H^{1/2}_0(\Gamma_1)'\) are given **Cauchy data** and \( f \in L^2(\Omega) \) is a known source term in the model.

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A solution of \((CP_{nl})\) is a \(H^1(\Omega)\)-distribution, which solves the weak formulation of the nonlinear elliptic equation \(P(u) = f\) in \(\Omega\) and further satisfies the Cauchy data at \(\Gamma_1\) in the sense of the trace operators. Notice that, if we know the Neumann (or Dirichlet) trace of \(u\) at \(\Gamma_2\), say \(q(u) u_\nu|_{\Gamma_2} = \varphi\), then \(u\) can be computed as the solution of a nonlinear mixed boundary value problem (BVP) in a stable way, namely

\[
P(u) = f, \quad u = g_1, \text{ at } \Gamma_1 \quad q(u) u_\nu = \varphi, \text{ at } \Gamma_2.
\]

Therefore, in order to solve \((CP_{nl})\), it is enough to consider the task of determining the Neumann trace of \(u\) at \(\Gamma_2\) (a distribution in \(H^{1/2}(\Gamma_2)\)).

**Remark 1.1** For simplicity of the presentation the boundary parts \(\Gamma_i\) are assumed to be connected. Using standard elliptic theory one can prove that the results in this article also hold without this assumption. Moreover, the theory derived here extends naturally to Cauchy problems defined at domains with \(\partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3\), where \(\Gamma_i\) are disjoint and some extra boundary condition (Dirichlet, Neumann, Robin, ...) is prescribed at \(\Gamma_3\).

**Remark 1.2** Let \(P\) be the linear elliptic operator defined in \(\Omega\) by

\[
P(u) := -\sum_{i,j=1}^{3} D_i (a_{i,j} D_j u),
\]

where the real functions \(a_{i,j} \in L^\infty(\Omega)\) are such that the matrix \(A(x) := (a_{i,j})^{d}_{i,j=1}\) satisfies \(\xi^t A(x) \xi > \alpha ||\xi||^2\), for all \(\xi \in \mathbb{R}^3\) and for a.e. \(x \in \Omega\). Here \(\alpha\) is some positive constant. The linear elliptic Cauchy problem corresponds to the problem \((CP_{nl})\) obtained when the operator \(P\) is substituted by \(P\) and the Neumann boundary condition is substituted by \(u_\nu|_{\Gamma_1} = g_2\). The linear version of \((CP_{nl})\) has been intensively investigated over the last years \([4, 5, 6, 7, 10, 11, 14, 16, 20, 21, 22, 24, 25]\).

Linear elliptic Cauchy problems were used by Hadamard in the 1920's as an example of (exponentially) ill-posed problem \([19]\). For linear elliptic operators with analytical coefficients, uniqueness of solutions is known for over half a century \([9]\). Moreover, as a straightforward argumentation with the Schwarz reflection principle shows, existence of solutions for arbitrary Cauchy data cannot be guaranteed \([18, 24]\). More recently, analytical features of linear elliptic Cauchy problems were investigated in \([4, 5, 10, 16, 24]\) and several numerical approaches were considered in \([4, 6, 7, 11, 14, 16, 20, 21, 22, 24, 25]\).

What concerns nonlinear elliptic Cauchy problems, analytical and numerical approaches can be found in \([23]\). Uniqueness of \(H^1(\Omega)\) solutions for \((CP_{nl})\) was proved in \([23, \text{Theorem } 2]\). The result in \([4, \text{Lemma } 2.1]\), which
guarantees existence of solutions of linear Cauchy problems for a dense subset of Cauchy data in $H^{1/2}(\Gamma_1) \times H^{1/2}_{\text{in}}(\Gamma_1)'$, can be extended for $(CP_{nl})$ (see Appendix).

Elliptic Cauchy problems arise in many industrial, engineering and biomedical applications including (A) Expansion of measured surface fields inside a body from partial boundary measurements [4]; (B) A classical thermostatics problem, which consists in recovering the temperature in a given domain when its distribution and the heat flux are known over the accessible region of the boundary [14]; (C) The analogous electrostatics case encountered in electric impedance tomography [4]; (D) An inverse problem related to corrosion detection [3, 25].

As a motivation for the specific problems analyzed in this article, we mention application (D) above. The detection of corrosion on an unaccessible boundary part of a domain leads to the inverse problem of identifying a piecewise constant function.

Our first main goal in this article is to derive convergence analysis for an iterative method of level-set type and investigate the application of this method to the exponentially ill-posed problem $(CP_{nl})$. The second main goal is to numerically investigate the application of the proposed method for solving three dimensional problems, where $\Gamma_2 \subset \partial \Omega$ is a 2D-manifold.

The introduction of level-set methods in the treatment of inverse problems was done by Santosa [27]. These methods are adequate to solve Cauchy problems in the special case where the Neumann trace of $u$ at $\Gamma_2$ is known a priori to satisfy $u_\nu|_{\Gamma_2} = \chi_D$, for some $D \subset \Gamma_2$. More recent approaches of level-set methods for inverse problems can be found in [8, 17, 25]. In [25] level-set type methods were used to approximate the solutions of linear elliptic Cauchy problems.

The manuscript is outlined as follows: In Section 2 we write the elliptic Cauchy problem $(CP_{nl})$ in the functional analytical framework of an (ill-posed) operator equation. In Section 3 we propose and analyze an iterative method of level-set type for $(CP_{nl})$ based on the approach followed in [17, 25]. Section 4 is devoted to numerical implementations. Some 3D experiments are provided, in order to illustrate the effectiveness of the iterative method considered in this article. In particular, we investigate how to improve performance by using a specially suited pre-conditioning strategy.
2 Functional analytical formulation

In this section we write the elliptic Cauchy problem \((CP_{nl})\) in the form of an operator equation in Hilbert spaces. In what follows we assume the coefficient function \(q\) in (1) to satisfy the following assumptions

\[\begin{align*}
A1) & \quad q \in C^\infty(\mathbb{R}); \\
A2) & \quad q(t) \in [q_{\text{min}}, q_{\text{max}}] \text{ for all } t \in \mathbb{R}, \text{ where } 0 < q_{\text{min}} < q_{\text{max}} < \infty.
\end{align*}\]

2.1 A linearization step

In the sequel the nonlinear problem \((CP_{nl})\) is transformed into a linear elliptic Cauchy problem, which is then reduced to a linear operator equation. The first step is to define the primitive of function \(q\)

\[Q(t) := \int_0^t q(s) \, ds\]

\((Q\) is strictly monotone increasing and therefore invertible). Notice that, given \(u \in L^2(\Omega)\) the function \(U := Q(u)\) is also in \(L^2(\Omega)\) and satisfies

\[-\Delta U = -\nabla \cdot (\nabla Q(u)) = -\nabla \cdot (q(u)\nabla u) = P(u)\, \text{.}\]

Moreover, \(U_\nu = q(u)u_\nu\) holds at \(\partial \Omega\). Thus, if \(u\) is the solution of \((CP_{nl})\) then \(U\) solves the linear Cauchy problem

\[(CP_l) \quad -\Delta U = f, \text{ in } \Omega \quad U = Q(g_1), \text{ at } \Gamma_1 \quad U_\nu = g_2, \text{ at } \Gamma_1.\]

Reciprocally, if problem \((CP_l)\) admits a solution, say \(U\), for the Cauchy data \((Q(g_1), g_2)\), it follows from \(Q' = q > 0\) (cf. assumption \(A2)\)), that \(u := Q^{-1}(U) \in H^1(\Omega)\) and solves problem \((CP_{nl})\). Summarizing, in order to obtain a solution for \((CP_{nl})\) it is necessary and sufficient to solve the linearized problem \((CP_l)\).

Next we consider the auxiliary problems

\[\begin{align*}
-\Delta w_a &= 0, \text{ in } \Omega & w_a &= 0, \text{ at } \Gamma_1 & (w_a)_\nu &= \varphi, \text{ at } \Gamma_2 \quad (2) \\
-\Delta w_b &= f, \text{ in } \Omega & w_b &= Q(g_1), \text{ at } \Gamma_1 & (w_b)_\nu &= 0, \text{ at } \Gamma_2 \quad (3)
\end{align*}\]

in order to define the function \(z := (w_b)_\nu|_{\Gamma_1}\), and the operator

\[L : \varphi \mapsto (w_a)_\nu|_{\Gamma_1}.\]
It is straightforward to check that $\varphi = q(u) u\nu|\Gamma_2 (= U\nu|\Gamma_2)$ is the solution of $(CP_{nl})$ iff it is a solution of the operator equation

$$L\varphi = g_2 - z.$$  \hfill (5)

Notice that $z = z(g_1, f, q)$ can be computed a priori.

A more precise definition of $L$ as well as some regularity properties are investigated in Subsection 2.1.

2.2 Abstract functional analytical framework

We consider $(CP_{nl})$ in the form of the operator equation (5). Further we assume the Cauchy data to satisfy

$$(g_1, g_2) \in H^{1/2}(\Gamma_1) \times H^{1/2}_{00}(\Gamma_2)'$$  \hfill (6)

and the source term $f$ to be a distribution in $L^2(\Omega)$.

Due to the choice of $g_1$, $f$ and $q$, it follows from the elliptic theory [24, Theorem C.2] that the mixed BVP in (3) has a unique solution $v_b \in H^1(\Omega)$. Therefore, $z := (v_b)_\nu|\Gamma_1 \in H^{1/2}_{00}(\Gamma_1)'$ (cf. [24, Theorem A.4]) and the term $g_2 - z$ on the right hand side of (5) is a distribution in $H^{1/2}_{00}(\Gamma_1)'$.

The next result ensures that the linear operator $L$ in (4) is well defined under appropriate choice of spaces.

**Proposition 2.1** Let $\Omega \subset \mathbb{R}^3$ and $\Gamma_i$, $i = 1, 2$, be defined as in Section 1. The operator defined in (4) is a linear injective bounded map $L : L^{3/2}(\Gamma_2) \to H^{1/2}_{00}(\Gamma_1)'$.

**Sketch of the proof.** The linearity of $L$ is obvious. Since the boundary part $\Gamma_2$ is a 2D-manifold, it follows from the Sobolev embedding theorem [1] that the embedding $H^s(\Gamma_2) \subset L^p(\Gamma_2)$ is compact for $p < 2(1 - s)^{-1}$. In particular we have $H^{1/2}(\Gamma_2) \hookrightarrow L^p(\Gamma_2)$ for $p < 4$, from what follows

$$H^{1/2}_{00} \subset H^{1/2} \subset L^3 \quad \text{and} \quad L^{3/2} = [L^3]' \subset H^{-1/2} \subset [H^{1/2}_{00}]'$$

(recall that $H^s = H^s_0$ for $s \leq 1/2$). Then, given $\varphi \in L^{3/2}(\Gamma_2)$, it follows from the elliptic theory [24, Theorem C.2] that the mixed BVP in (2) has a unique solution $w_a \in H^1(\Omega)$ satisfying the a priori estimate

$$\|v_a\|_{H^1(\Omega)} \leq C_1 \|\varphi\|_{H^{1/2}_{00}(\Gamma_2)}'.$$
where \( C_1 = C_1(\Omega, \Gamma_2) > 0 \). Moreover, from the continuity of the Neumann trace operator \( \gamma_{N,1} : H^1(\Omega) \ni v \mapsto v \nu |_{\Gamma_1} \in H^{1/2}_{00}(\Gamma_1)' \) (cf. [24, Theorem A.4]), follows
\[
\| L \varphi \|_{H_{00}^{1/2}(\Gamma_1)'} \leq C_2 \| v_a \|_{H^1(\Omega)} \leq C_3 \| \varphi \|_{L^{3/2}(\Gamma_2)},
\]
proving the boundedness of \( L \). It remains to prove the injectivity. Notice that, if \( L \varphi = 0 \) then \( w_a \) in (2) satisfies: \( \Delta w_a = 0 \) in \( \Omega \), \( w_a = (w_a)_{\nu} = 0 \) at \( \Gamma_1 \). Thus, \( \varphi = 0 \) follows from the uniqueness of weak solution for \((CP)\) [24, Theorem D.3].

Summarizing, if the Cauchy data is given as in (6) and assumptions \( A1), A2 \) hold, then problem \((CP_{nl})\) can be stated in the form of equation (5), where \( L \) is the linear continuous and injective operator
\[
L : L^{3/2}(\Gamma_2) \to H_{00}^{1/2}(\Gamma_1)',
\]
defined in (4).

**Remark 2.2** The choice of the space \( L^{3/2}(\Gamma_2) \) in Proposition 2.1 is non standard. More natural would be the choice \( H_{00}^{1/2}(\Gamma_2)' \). This point will become clear when we introduce the level-set method in Section 3 (Lemma 3.3).

### 2.3 A remark on noisy Cauchy data

If only corrupted noisy data \((g^\delta_1, g^\delta_2)\) are available for problem \((CP_{nl})\), we assume the existence of a consistent Cauchy data \((g_1, g_2)\) satisfying (6) such that
\[
\| g_1 - g^\delta_1 \|_{L^2(\Gamma_1)} + \| g_2 - g^\delta_2 \|_{L^2(\Gamma_1)} \leq \delta.
\]
Since \( z \) in (5) depends continuously on \( g_1 \) in the \( H^{1/2}(\Gamma_1) \) topology, a natural question arrises:

Q) Is it possible to obtain from measured data \((g^\delta_1, g^\delta_2)\) satisfying (8), a corresponding \( z^\delta \in H_{00}^{1/2}(\Gamma_1)' \) such that \( \| z - z^\delta \|_{H_{00}^{1/2}(\Gamma_1)'} \to 0 \) as \( \delta \to 0 \)?

The next Lemma gives a positive answer to this question.

**Lemma 2.3** Let the noisy Cauchy data be given as in (8), where \( g_1 \in H^s(\Gamma_1) \) for some \( s > 1/2 \). Then \((CP_{nl})\) reduces to the operator equation
\[
L \varphi = g^\delta_2 - z^\delta,
\]
where the right hand side satisfies
\[
\| (g_2 - z) - (g^\delta_2 - z^\delta) \|_{H_{00}^{1/2}(\Gamma_1)'} \leq h(\delta).
\]
Here \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) is a function satisfying \( \lim_{\delta \to 0} h(\delta) = 0 \).
Sketch of the proof. Notice that \( \|g_2 - g_\delta^2\|_{H^{1/2}_0(\Gamma_1)'} \leq \|g_2 - g_\delta^2\|_{L^2(\Gamma_1)} \). The key argument to construct \( z_\delta \) and the function \( h \) is the existence of a continuous smoothing operator \( S : L^2(\Gamma_1) \to H^{1/2}(\Gamma_1) \) and of a function \( \mu : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \lim_{\delta \to 0} \mu(\delta) = 0 \), such that \( \|g_1 - S(g_\delta^1)\| \leq h(\delta) \). For details see [14, Section 4.2].

Lemma 2.3 It will be used in Section 3 in the derivation of some classical regularization theory results.

3 Level-set approximations

In this section we investigate a level-set type method for \((CP_{nl})\). Our approach is based on [17]. In what follows we shall consider the functional analytical framework for \((CP_{nl})\) discussed in Subsection 2.2.

3.1 Level-set approach and constrained optimization

The standard level-set approach uses the assumption that the solution \( \varphi \) of (5) is piecewise constant, taking only one of two possible values. For simplicity, we assume that \( \varphi \) is the characteristic function \( \chi_D \) of a subdomain \( D \subset \subset \Gamma_2 \). Next we introduce a function \( \phi : \Gamma_2 \to \mathbb{R} \), in such a way that \( \varphi \) can be represented by a level-set of \( \phi \)

\[
\varphi(x) = \chi_D(x) = 1 \iff x \in D = \{ x \in \Gamma_2; \, \phi(x) \geq 0 \}.
\]

Under this assumption, the Cauchy problem (5) can be stated in the form of the least-square problem

\[
\min_{\phi \in H^1(\Gamma_2)} \| L(H(\phi)) - (g_2^\delta - z^\delta) \|_Y^2,
\]

where \( Y := H^{1/2}_0(\Gamma_1)' \) and \( H \) the Heaviside projector.

The level-set method discussed here corresponds to a continuous evolution of the level-set function \( \phi \) for an artificial time \( t \). This evolution is motivated by the minimization of the Tikhonov functional

\[
\mathcal{F}_\alpha(\phi) := \| L(H(\phi)) - (g_2^\delta - z^\delta) \|_Y^2 + \alpha \left[ \beta |H(\phi)|_{BV} + \| \phi - \phi_0 \|_{H^1}^2 \right],
\]

based on \( TV-H^1 \)-penalization for the least-square functional in (10). Here \( \alpha > 0 \) plays the role of a regularization parameter and \( \beta > 0 \) is a scaling factor.
Since $H$ is discontinuous (considered as an operator from $H^1$ to $L^{3/2}$), one cannot prove that the Tikhonov functional in (11) attains a minimizer. In order to guarantee existence of minimizers for $F_\alpha$, it is necessary to use a generalized minimizer concept. With this in mind we define

**Definition 3.1** Let the boundary part $\Gamma_2 \subset \partial \Omega$ be defined as in Section 1.

i) A pair of functions $(\psi, \phi) \in L^\infty(\Gamma_2) \times H^1(\Gamma_2)$ is called admissible if there exists a sequence $\{\phi_k\}_{k \in \mathbb{N}}$ in $H^1(\Gamma_2)$ such that $\phi_k \rightharpoonup \phi$ with respect to the $L^2(\Gamma_2)$-norm, and there exists a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ of positive numbers converging to zero such that $H_{\varepsilon_k}(\phi_k) \rightharpoonup \psi$ in $L^{3/2}(\Gamma_2)$.

ii) The set of admissible pairs is defined by

$$Ad := \{(\psi, \phi) \in L^\infty(\Gamma_2) \times H^1(\Gamma_2) ; \exists \{\phi_k\} \in H^1 \text{ and } \{\varepsilon_k\} \in \mathbb{R}^+ \text{ s.t. } \lim_{k \to \infty} \varepsilon_k = 0, \lim_{k \to \infty} \|\phi_k - \phi\|_{L^2} = 0, \lim_{k \to \infty} \|H_{\varepsilon_k}(\phi_k) - \psi\|_{L^{3/2}} = 0\}.$$

iii) The functional $F_\alpha(\psi, \phi)$ is defined on $Ad$ by

$$F_\alpha(\psi, \phi) := \|L\psi - (g_2^\delta - z^\delta)|_Y^2 + \alpha \rho(\psi, \phi), \quad (12)$$

where $\rho(\psi, \phi) := \inf_{\{\phi_k\}, \{\varepsilon_k\}} \liminf_{k \to \infty} \{2\beta |H_{\varepsilon_k}(\phi_k)|_{BV} + \|\phi_k - \phi_0\|_{H^1}^2\}$, the infimum being taken with respect to all sequences $\{\phi_k\}_{k \in \mathbb{N}}$ and $\{\varepsilon_k\}_{k \in \mathbb{N}}$ characterizing $(\psi, \phi)$ as an element of $Ad$.

iv) A generalized minimizer of $F_\alpha(\phi)$ is a minimizer of $F_\alpha(\psi, \phi)$ on $Ad$.

**Remark 3.2** A consequence of the definition above is the fact that $F_\alpha$ is no longer considered as a functional on $H^1$, but as a functional defined on the $w$-closure of the graph of $H$, contained in $BV \times H^1$.

Another consequence is that the penalization term in (11) can now be interpreted as a functional $\rho : Ad \to \mathbb{R}^+$.  

### 3.2 Convergence analysis

In order to prove coerciveness and weak lower semi-continuity of $\rho$, the assumption that $L$ is a continuous operator on a $L^{3/2}$-space is crucial (see Proposition 2.1). These properties of $\rho$ are the main arguments needed to

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1. Given $\varepsilon > 0$, the functions $H_\varepsilon$ are defined by

$$H_\varepsilon(t) := \begin{cases} 0 & \text{for } t < -\varepsilon, \\ 1 + t/\varepsilon & \text{for } -\varepsilon \leq t \leq 0, \\ 1 & \text{for } t \geq 0. \end{cases}$$
prove existence of a generalized minimizer \((\overline{\psi}_\alpha, \overline{\phi}_\alpha)\) of \(F_\alpha\) in \(Ad\), as we shall see next:

**Lemma 3.3** Let the boundary part \(\Gamma_2 \subset \partial \Omega\) be defined as in Section 1. The following assertions hold true:

i) The semi-norm \(\cdot\)_{BV}\) is weakly lower semi-continuous with respect to \(L^{3/2}\)-convergence;

ii) \(BV(\Gamma_2)\) is compactly embedded in \(L^{3/2}(\Gamma_2)\).

*Proof*. For (i) see [2, Section 2.3.2]. For (ii) see [15, Section 5.2.1]. \(\square\)

**Theorem 3.4** Let the functionals \(\rho, F_\alpha\) and the set \(Ad\) be defined as above. The following assertions hold true:

i) The functional \(\rho(\psi, \phi)\) is coercive on \(Ad\);

ii) The functional \(\rho(\psi, \phi)\) is weakly lower semi-continuous on \(Ad\);

iii) The functional \(F_\alpha(\psi, \phi)\) attains a minimizer on \(Ad\).

*Proof*. (i) Let \((\psi, \phi) \in Ad\). Then, there exist sequences \(\{\phi_k\}_{k \in \mathbb{N}}\) and \(\{\varepsilon_k\}_{k \in \mathbb{N}}\) as in Definition 3.1 (i). Thus, \(\|\phi - \phi_0\|^2_{H^1} \leq \liminf_{k \to \infty} \|\phi_k - \phi_0\|^2_{H^1}\). Moreover, Lemma 3.3 implies \(\|\psi\|_{BV} \leq \liminf_{k \to \infty} |H_{\varepsilon_k}(\phi_k)|_{BV}\). Therefore,

\[
2\beta |\psi|_{BV(\Gamma_2)} + \|\phi - \phi_0\|^2_{H^1(\Gamma_2)} \leq \rho(\psi, \phi).
\]

(ii) Follows from Lemma 3.3 and the weak lower semi-continuity of the \(H^1\)-norm.

(iii) Since \((0, -1) \in Ad\), then \(Ad \neq \emptyset\) and \(\inf F_\alpha < \infty\). Let \((\psi_k, \phi_k) \in Ad\) be a minimizing sequence for \(F_\alpha\), i.e. \(F_\alpha(\psi_k, \phi_k) \to \inf F_\alpha\) as \(k \to \infty\). This fact implies the boundedness of \(\rho(\psi_k, \phi_k)\). Item (i) above implies the boundedness of both sequences \(\|\phi_k - \phi_0\|^2_{H^1}\) and \(|\psi_k|_{BV}\). From the compactness of the embedding \(H^1 \hookrightarrow L^2\) and Lemma 3.3 (ii) we can extract subsequences (again denoted by \(\{\psi_k\}\) and \(\{\phi_k\}\)) such that

\[
\psi_k \to \psi \text{ in } BV, \quad \psi_k \to \psi \text{ in } L^{3/2}, \quad \phi_k \to \phi \text{ in } H^1, \quad \phi_k \to \phi \text{ in } L^2,
\]

for some \((\psi, \phi) \in BV(\Gamma_2) \times H^1(\Gamma_2)\). Now, arguing with (7) and item (ii) above, one obtains

\[
\inf F_\alpha = \lim_{k \to \infty} F_\alpha(\psi_k, \phi_k) = \lim_{k \to \infty} \{\|L\psi_k - (g_2^\delta - z^\delta)\|_{L^2_Y}^2 + \alpha \rho(\psi_k, \phi_k)\} \\
\geq \liminf_{k \to \infty} \{\|L\psi_k - (g_2^\delta - z^\delta)\|_{L^2_Y}^2 + \alpha \rho(\psi_k, \phi_k)\} \\
\geq \|L\psi - (g_2^\delta - z^\delta)\|_{L^2_Y}^2 + \alpha \rho(\psi, \phi) = F_\alpha(\psi, \phi).
\]

It remains to prove that \((\psi, \phi) \in Ad\). This is done analogously as in the final part of the proof of [17, Theorem 2.9]. \(\square\)
Remark 3.5 If the Cauchy data \((g_1, g_2)\) is consistent, i.e. \(\delta = 0\) in (8), the existence of a minimum norm solution \((\psi^\dagger, \phi^\dagger)\) can be proved, i.e. an element \((\psi^\dagger, \phi^\dagger)\in Ad, such that \(L(\psi^\dagger) = g_2 - z, and\)

\[ \rho(\psi^\dagger, \phi^\dagger) = \inf \{ \rho(\psi, \phi); (\psi, \phi) \in Ad \text{ and } L(\psi) = g_2 - z \}. \]

The proof of this result follows the lines of the proof of [17, Theorem 2.10].

The classical analysis of Tikhonov type regularization methods [13] do apply to functional \(F_\alpha\), as we shall see next.

Theorem 3.6 (Convergence) Let the Cauchy data \((g_1, g_2)\) be consistent. Moreover, Let \(\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) be a function satisfying \(\lim_{\delta \rightarrow 0} \alpha(\delta) = 0 \) and \(\lim_{\delta \rightarrow 0} \delta^2 \alpha(\delta) = 0\). Given a sequence \(\delta_k \rightarrow 0\) and \(\{ (g_{\delta_k}^1, g_{\delta_k}^2) \}_{k}\) corresponding noisy data satisfying (8), the generalized minimizers \((\psi_k, \phi_k)\) of \(F_{\alpha(\delta_k)}\) converge in \(L^{3/2} \times L^2\) to a generalized minimizer \((\overline{\psi}, \overline{\phi})\in Ad\) of \(F_\alpha\).

Proof. The proof uses classical techniques from the analysis of Tikhonov regularization methods [13] and thus omitted.

3.3 Stabilized approximation

We conclude this section with a result that guarantees the efficiency of a numerical approximation scheme for solving (5). Indeed, we prove that the generalized minimizers of the functional \(F_\alpha\) defined in (12) can be approximated by minimizers of the stabilized functional

\[
F_{\alpha, \varepsilon}(\phi) := \| L(H_\varepsilon(\phi)) - (g_2 - z) \|^2_Y + \alpha \left[ \beta \| H_\varepsilon(\phi) \|_{BV} + \| \phi - \phi_0 \|^2_{H^1} \right] \tag{13}
\]

in the following sense:

Theorem 3.7 If \(\phi_{\alpha, \varepsilon}\) are minimizers of \(F_{\alpha, \varepsilon}\) then, given a sequence \(\varepsilon_k \rightarrow 0^+\), there exists a subsequence \((H(\phi_{\alpha, \varepsilon}), \phi_{\alpha, \varepsilon})\) converging in \(L^{3/2}(\Gamma_2) \times L^2(\Gamma_2)\) and the limit minimizes \(F_\alpha\), in Ad.

Sketch of the proof. The minimizers \(\phi_{\alpha, k}\) of \(F_{\alpha, k}\) are uniformly bounded in \(H^1\). Moreover, \(H_{\varepsilon_k}(\phi_{\alpha, k})\) is uniformly bounded in BV. Then (up to subsequences) these sequences converge strongly in \(L^{3/2} \times L^2\) to a limit \((\tilde{\psi}, \tilde{\phi})\in BV \times H^1\) (notice that from this convergence follows \((\tilde{\psi}, \tilde{\phi})\in Ad\)). In order to prove that \((\tilde{\psi}, \tilde{\phi})\) minimizes \(F_\alpha,\) one argues with (7) and Theorem 3.4.

The existence of minimizers of \(F_{\alpha, \varepsilon}\) in \(H^1(\Gamma_2)\) still has to be cleared:
Lemma 3.8 For any $\phi_0 \in H^1(\Gamma_2)$ the functional $F_{\alpha,\varepsilon}$ in (13) attains a minimizer.

Proof. Notice that a minimizing sequence $\{\phi_k\}$ for $F_{\alpha,\varepsilon}$ is bounded in $H^1(\Gamma_2)$. Therefore, up to a subsequence, we have $\phi_k \rightharpoonup \phi$ in $H^1$ and $\phi_k \to \phi$ in $L^2$ for some $\phi_{\alpha,\varepsilon} \in H^1(\Gamma_2)$. On the other hand,

$$||H_\varepsilon(\phi_k) - H_\varepsilon(\phi_{\alpha,\varepsilon})||_{L^{3/2}(\Gamma_2)} \leq \varepsilon^{-1} \text{meas}(\Gamma_2)^{1/6} ||\phi_k - \phi_{\alpha,\varepsilon}||_{L^2(\Gamma_2)} \to 0,$$

and from Lemma 3.3 (i) follows $|H_\varepsilon(\phi_{\alpha,\varepsilon})|_{BV} \leq \liminf_k |H_\varepsilon(\phi_k)|_{BV}$. The lemma follows now from (7) and the weak lower semi-continuity of the $H^1$-norm.

This relation between the minimizers of $F_\alpha$ and $F_{\alpha,\varepsilon}$ is the starting point for the derivation of a numerical method. This is our main goal in the next section (see Subsection 4.3).

4 Numerical realization and experiments

In this section we illustrate the usability of our approach by numerical experiments. After introducing our model problem, we shortly discuss its discretization and then report on details concerning the implementation of the level-set method in Section 3. We conclude with presenting results of some numerical tests.

4.1 The model problem and its discretization

Let us consider the following nonlinear Cauchy problem: For $a > 0$ let $\Omega := (0,1) \times (0,1) \times (0,a)$ and the boundary $\partial \Omega$ be composed of three parts $\partial \Omega = \Gamma_0 \cup \Gamma_L \cup \Gamma_a$ with

$$\Gamma_0 := (0,1)^2 \times \{0\}, \quad \Gamma_a := (0,1)^2 \times \{a\}, \quad \text{and} \quad \Gamma_L := \partial \Omega \setminus \Gamma_0 \cup \Gamma_a.$$ 

We consider the solution of the Cauchy problem $L\varphi = g^\delta$ with forward operator $L$ defined by $L(\varphi) := q(u)u_\nu|_{\Gamma_0}$ and $u$ being the solution of the nonlinear BVP

$$-\nabla \cdot (q(u)\nabla u) = 0, \text{ in } \Omega \quad u = 0, \text{ on } \Gamma_0 \cup \Gamma_L \quad q(u)u_\nu = \varphi \text{ on } \Gamma_a, \quad (14)$$

with nonlinear coefficient $q(u) = 1 + u^2$.

The choice $u = 0$ on the lateral boundary $\Gamma_L$ above is done to simplify the calculations (see Remark 1.1). For solution of the nonlinear mixed BVP

$$\nabla \cdot (q(u)\nabla u) = 0, \text{ in } \Omega \quad u = 0, \text{ on } \Gamma_0 \cup \Gamma_L \quad q(u)u_\nu = \varphi \text{ on } \Gamma_a, \quad (14)$$

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with nonlinear coefficient $q(u) = 1 + u^2$. 

The choice $u = 0$ on the lateral boundary $\Gamma_L$ above is done to simplify the calculations (see Remark 1.1). For solution of the nonlinear mixed BVP

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with nonlinear coefficient $q(u) = 1 + u^2$.
(14) we consider a finite difference discretization. In order to cope with the nonlinearity, we propose a simple fix-point iteration: let $P(u)$ denote the stiffness Matrix of the operator $-\nabla \cdot (q(u)\nabla)$ and $b$ denote the right hand side of the discretization resembling the non-homogeneous Neumann data. For computing a sequence of iterates we use the schema

$$P(u_n)u_{n+1} = b,$$  \hspace{1cm} (15)

which is stopped as soon as the norm of the residual $r_k = b - P(u_n)u_n$ has reached a required tolerance of $10^{-8}$. In each step of the iteration, the linearized systems (15) are solved by a preconditioned conjugate gradient method. Throughout our numerical tests, the fix-point iteration converged within less then 10 iteration to the required tolerance.

### 4.2 Linearization and an alternative discretization

Following the linearization procedure outlined in Subsection 2.1 we first transform the nonlinear Cauchy problem (14) into a linear one by setting $U(x, y, z) = Q(u(x, y, z))$ with $Q(u) = \int_0^u q(v)dv = u + \frac{1}{2}u^3$ being the primitive of $q$. The linearized Cauchy problem then reads

$$L\varphi = g^\delta$$  \hspace{1cm} (16)

(notice that $z$ in (5) vanishes due to the particular choice of $g_1 = f = 0$ and $q(u) = 1 + u^2$) where the operator $L$ is now defined by $L\varphi = U_\nu$ and $U$ solves the system

$$-\Delta U = 0, \text{ in } \Omega \hspace{1cm} U = 0, \text{ on } \Gamma_0 \cup \Gamma_L \hspace{1cm} U_\nu = \varphi \text{ on } \Gamma_a.$$  

For the numerical solution of the linearized BVP, we consider a method based on Fourier expansions. Let $\varphi_{m,n}$ denote the Fourier coefficients of a function $\varphi$ with respect to the expansion

$$\varphi(x, y) = \sum_{m,n} \sin(m\pi x) \sin(n\pi y).$$

The solution $U$ of the above BVP is then given by

$$U(x, y, z) = \sum_{m,n} U_{m,n} \sin(m\pi x) \sin(n\pi y) \sinh(w_{m,n}\pi z),$$

with Fourier coefficients $U_{m,n} = \varphi_{m,n}/(w_{m,n}\pi \cosh(w_{m,n}\pi a))$ and $w_{m,n} := \sqrt{m^2 + n^2}$. Consequently, the forward operator $L$ has the Fourier series
representation

\[(L\varphi)(x, y, z) = \sum_{m,n} g_{m,n} \sin(m\pi x) \sin(n\pi y) \sinh(w_{m,n} \pi z)\]

with

\[g_{m,n} := -\frac{1}{\cosh(w_{m,n} \pi a)} \varphi_{m,n} =: A_{m,n}^{-1}(m,n)\varphi_{m,n}.\]

Here the numbers \(A_{m,n}^{-1} \leq 1\) denote damping factors of the Fourier components of \(\varphi\). The formal solution \(\varphi^\delta\) of the Cauchy problem (16) is then given by

\[\varphi^\delta(x, y, z) = \sum_{m,n} g_{m,n} \sin(m\pi x) \sin(n\pi y) \sinh(w_{m,n} \pi a).\]

Errors in the \((m,n)\)-th Fourier coefficient of the data are amplified by a factor \(A_{m,n} = \cosh(w_{m,n} \pi a)\), showing that the inverse problem (16) is exponentially ill-posed for \(a \neq 0\).

For discretization of the linearized problem we propose a discrete Fourier transform (DFT) and obtain a discretized forward operator \(L_N\) defined by

\[(L_N\varphi)(x, y, z) = \sum_{m,n=1}^N g_{m,n} \sin(m\pi x) \cos(n\pi x) \sinh(w_{m,n} \pi z).\]

The discrete Fourier coefficients of a function \(g\) are given by

\[g_{m,n} = \frac{2}{N} \sum_{i,j=1}^N \sin(m\pi x_i) \sin(n\pi y_j) g(x_i, y_j).\]

The action of the forward operator \(L\) can thus be computed efficiently by fast Fourier transforms and multiplication in the frequency domain.

4.3 Implementation of the level-set approach

Let us shortly discuss how minimizers of the functional \(F_\alpha\) can be found numerically. As shown in Subsection 3.3, generalized minimizers of (12) can be approximated by minimizers of the stabilized functional \(F_{\alpha,\varepsilon}\) defined in (13). In our numerical experiments, we choose

\[H_\varepsilon(x) = 2^{-1}[\text{erf}(\frac{x}{\varepsilon}) + 1],\]
where \( \text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \) denotes the error function.\(^2\) Let \( \phi_{\alpha,\varepsilon_n} \) be a minimizer of \( F_{\alpha,\varepsilon_n} \) for a sequence \( \varepsilon_n \to 0^+ \). Then one can find a subsequence \( (H_{\varepsilon}(\phi_{\alpha,\varepsilon_n}), \phi_{\alpha,\varepsilon_n}) \) converging in \( L^{3/2}(\Gamma_2) \times L^2(\Gamma_2) \) and the limit minimizes \( F_{\alpha} \) in \( \text{Ad} \). In the sequel we only discuss how to find minimizers of the stabilized functional.

For the derivation of a method, let us start from the necessary first order conditions for a minimum of (13)

\[
0 = -H_{\varepsilon}'(\phi)L(H_{\varepsilon}(\phi))^*[L(H_{\varepsilon}(\phi)) - g^\delta] \\
+ \alpha[\beta H_{\varepsilon}'(\phi)\nabla \cdot (\nabla H_{\varepsilon}(\phi)/|\nabla H_{\varepsilon}(\phi)|)] + (I - \Delta)(\phi - \phi_0) =: R_{\alpha,\varepsilon}(\phi). \tag{17}
\]

Here \( L^* \) denotes the adjoint of the operator \( L \) with respect to the \( H^{1/2}_0 - H^{-1/2}_0 \) duality pairing.

For finding a solution of (17) a simple fixed point iteration was proposed in [17]. Here, we use a different approach based on the ideas of the Gauß-Newton method. For \( \beta = 0 \), we define the update \( \delta \phi_k = \phi_{k+1} - \phi_k \) by

\[
[H_{\varepsilon}'(\phi_k)L(H_{\varepsilon}(\phi_k))^*L(H_{\varepsilon}(\phi_k))H_{\varepsilon}'(\phi_k) + \alpha(I - \Delta)]\delta \phi_k = -R_{\alpha,\varepsilon}(\phi_k). \tag{18}
\]

In case \( \beta \neq 0 \), one can add an additional term accounting for the BV regularization to the Gauß-Newton matrix. In our numerical experiments however, we could always set \( \beta = 0 \). After space discretization, the linear systems (18) can be solved, e.g., by the conjugate gradient method.

The iteration (18) can be stopped as soon as the norm of \( R_{\alpha,\varepsilon} \) is sufficiently small. The parameter \( \alpha \) can then be chosen by a discrepancy principle, i.e., one solves the minimization problem (13) for a sequence of decreasing values \( \alpha_n \). Arguing that the minima for different values of \( \alpha \) will be close together and that the residual \( R_{\alpha_{n+1},\varepsilon} \) will be reduced sufficiently by only one step of the iteration when started at the minimizer for \( \alpha_n \), one can alternatively replace the two nested iterations of determining the optimal \( \alpha \) and finding the minimizer by a single iteration. The resulting scheme corresponds to the iteratively regularized Gauß-Newton method [13]. In algorithmic form, we obtain for our test problem

1. choose \( \phi_0 \in H^1(\Gamma_2) \) and set \( k = 0 \);
2. loop\(^2\)

---

\(^2\)This definition of the operator \( H_{\varepsilon} \) is slightly different from the one given in Subsection 3.1. It is worth noticing that the theoretical results derived in Section 3 also hold with this definition.
(a) evaluate the residual \( r_k := L(H_\varepsilon(\phi_k)) - g^\delta \); the action of \( L \) on \( H_\varepsilon(\phi_k) \) is given by \( L(H_\varepsilon(\phi_k)) = (U_k)_\nu|_{\Gamma_0} \), where \( U_k \) solves
\[
\Delta U_k = 0, \text{ in } \Omega \\
(U_k)|_{\Gamma_0} = 0, \quad (U_k)_\nu|_{\Gamma_a} = H_\varepsilon(\phi_k), \quad U_k|_{\Gamma_L} = 0;
\]
(b) if \( \|r_k\| < \tau \delta \) stop;
(c) evaluate \( W_k := L(H_\varepsilon(\phi_k))^* r_k \); here \( L(H_\varepsilon(\phi_k))^* r_k = -V_k|_{\Gamma_a} \), where \( V_k \) solves the problem
\[
\Delta V_k = 0, \text{ in } \Omega \\
V_k|_{\Gamma_0} = r_k, \quad (V_k)_\nu|_{\Gamma_a} = 0, \quad V_k|_{\Gamma_L} = 0;
\]
(d) set \( Z_k := H_\varepsilon'(\phi_k) W_k + \alpha_k (I - \Delta) \phi_k \);
(e) compute the update \( \delta \phi_k \) by solving the linearized system
\[
[A_k^* A_k + \alpha_k (I - \Delta)] \delta \phi_k = -Z_k, \quad A_k := H_\varepsilon'(\phi_k) L(H_\varepsilon(\phi_k))
\]
with the method of conjugate gradients, where the action of the operators \( L \) and \( L^* \) is defined as above.
(f) update the level-set function \( \phi_{k+1} = \phi_k + \delta \phi_k \) and set \( k = k + 1 \);

Note that calculating \( Z_k = R_{\alpha,\varepsilon}(\phi_k) \) as well as each step of the conjugate gradient method requires the solution of two mixed BVPs. As outlined above, each solution can be performed by two fast Fourier transforms and a multiplication in the Fourier domain for our test examples.

4.4 First numerical experiment: ”almost” exact data

As data for the linearized Cauchy problem we consider the ones simulated by the finite difference method for the nonlinear problem. This justifies the title of this subsection.

In order to estimate the discretization error, we solve the linearized forward problem with the finite difference and the Fourier transform based method and take \( \delta := \|g_{DFT} - g_{FD}\| \) as a measure for the noise level. Here, \( g_{DFT} \) and \( g_{FD} \) denote the data generated by the two different methods. We also compared the data simulated by the finite difference method with the
ones generated by the DFT method with \( N \) replaced by \( 2N \), which gave similar values for the noise level \( \delta \). The data are then additionally perturbed by random noise of size \( \delta \).

In this numerical test, we set \( a = 1/\pi \) in the model problem of Subsection 4.1 and aim to reconstruct a binary valued coefficient (the unknown Neumann data) possessing several – large/small, convex/concave, round/edgy – features, cf. Figure 1 (a). The corresponding Cauchy data is \((0, g)\), where \( g \) is the measured heat flux (Neumann data at \( \Gamma_0 \)) depicted in Figure 1 (b).

The data \( g \) are generated by a finite difference solution of the nonlinear Cauchy problem (14) on a \( 100 \times 100 \times 100 \) grid. For solution of the inverse problem, we apply the Gauß-Newton level-set method introduced in Subsection 4.3 and use the method based on Fourier transforms discussed in Subsection 4.2.

Throughout our numerical experiments we use \( \varepsilon = 0.1 \). As initial level-set function we choose the parabola \( \phi_0(x, y) = 0.1^2 - (x - 0.5)^2 + (y - 0.5)^2 \). The initial zero level-set hence is a circle with center \((0.5, 0.5)\) and radius 0.1 (see top-left picture in Figure 2). The evolution of the 0.5-level-sets of the iterates \( H_\varepsilon(\phi_k) \) is shown in Figure 2. The corresponding evolution of the level-set functions \( \phi_k \) is depicted in Figure 3.

**Remark 4.1** In the previous calculations we set \( a = 1/\pi \). Taking \( a \) to be larger, the ill-posedness of the inverse problem is increased. For \( a \sim 1 \), the numerical error introduced by the gap between the finite difference solution for the nonlinear problem and the Fourier transform based solution of the

---

![Figure 1: Setup of the first numerical experiment: (a) True solution (Neumann data at \( \Gamma_a \)); (b) Measured heat flux (Neumann data at \( \Gamma_0 \)).](image)
linearized problem – for our discretization we have $\delta = 2.7 \times 10^{-4}$ – is too large to allow for a reasonable reconstruction. In order to achieve similar results as the ones reported for the problem with $a = 1$ the noise level has to be reduced to $10^{-10}$.

In Figure 4 we show the approximations obtained with our iterative method for different values of $a \in [0.1, 1]$. In all cases the same discretization level and noise level were used, and the stop criteria was reached after after 100 steps. Notice that, as the value of $a$ increases, the accuracy of the reconstruction deteriorates. As observed above, an accurate reconstruction can only be obtained if the increase of $a$ is accompanied by a corresponding decrease of the noise level and grid refinement.

We conclude this first experiment presenting a comparison between our method and the iterated Tikhonov method. Its worth noticing that the iterated Tikhonov method corresponds to the choice $H_{\epsilon}(x) = x$, i.e. no projection. The approximation obtained after 100 iterative steps and its corresponding 0.5-level-set is shown in Figure 5. Comparing the results in the bottom-right picture in Figure 2 with the results in Figure 5 one notices that method is clearly more advantageous.

### 4.5 Second numerical experiment: noisy data

In this second experiment we consider the same basic setup as in Subsection 4.4. This time however, we artificially introduce noise to the Cauchy data $g$ shown in Figure 1 (b) and aim to solve (16) with Cauchy data $(0, g^\delta)$.

In a first test, 1% random noise is added to the 'exact' data $g$ and the iterative scheme described in Subsection 4.3 is applied. In a second test, the data is perturbed with 5% random noise. The stop criteria is reached after a small number of iterations. The obtained results are depicted in Figure 6. As expected, the reconstruction becomes more and more unstable as the noise level increases. Fine structures can no longer be identified as the noise level increases. However, some large structures and basic features of the solution can still be recovered even in the presence of high levels of noise, what is unusual for exponentially ill-posed problems of this kind.

It is worth noticing that the preconditioning strategy tremendously improves the performance of the level-set method introduced in [17]. In [17, 25] this level-set method was implemented for exponentially ill-posed problems and several hundreds of iterations were needed to reach the stop criteria, while here only a few iterations are required. For further discussion on this issue we refer to [12].

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5 Final remarks and conclusions

In this article an iterative method of level-set type for solving nonlinear elliptic Cauchy problems in 3D is considered. A framework for the level-set approach is established and convergence analysis is developed (monotony, convergence, stability results).

Further we discuss the numerical realization of the proposed level-set method. Different numerical experiments illustrate relevant features of the method, like: rates of convergence, adaptability to identify non-connected inclusions, robustness with respect to noise.

The numerical method analyzed in this article can be extended in a straightforward way to arbitrary elliptic Cauchy problems possessing a solution with similar structure, i.e. whenever the assumption that \( q(u)u_\nu|_{\Gamma_2} \) is a piecewise constant function assuming one of only two possible values (not necessarily zero and one) is valid. The proposed method relates to evolution flows of Hamilton-Jacobi type.

Most of the analysis presented in Section 3 was formulated in [17] for operators \( L \) continuous in the \( L^1 \)-topology. We used Lemma 3.3 to adapt the convergence results in [17] to operators that are continuous in the \( L^{3/2} \)-topology. This allowed the application and analysis of this level-set type method for \((\text{CP}_{nl})\). It is worth noticing that Lemma 3.3 still holds for \( L^p \) with \( 1 \leq p < 2 \). Therefore, the analytical results in Section 3 can be extended to any linear inverse problem modeled by operators continuous in the \( L^p \)-topology.

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Appendix An existence result for \((\text{CP}_{nl})\)

Proposition A.1 Let \( \Omega \subset \mathbb{R}^3 \) and \( \Gamma_1 \) be defined as in Section 1. Moreover, let the operator \( \mathcal{P} \) be defined as in (1), where \( q : \mathbb{R} \to [q_{\min}, q_{\max}] \subset (0, \infty) \) is a \( C^\infty \)-function. There exists a dense subset \( M \subset H^{1/2}(\Gamma_1) \times H^{1/2}_{00}(\Gamma_1)' \) such that the nonlinear problem

\[
\text{The results in [17] cannot be directly applied to (CP), since } L^1 \text{ is not embedded in } [H^{1/2}_{00}]'. \text{ See Subsection 2.2.}
\]
(CP) \( P(u) = 0 \), in \( \Omega \hspace{1cm} u = g_1 \), at \( \Gamma_1 \hspace{1cm} q(u) u_\nu = g_2 \), at \( \Gamma_1 \),
is consistent for \((g_1, g_2) \in M\).

**Proof.** From Subsection 2.1 we know that problem \((CP_{nl})\) admits a solution \( u \) for the Cauchy data \((g_1, g_2)\) iff the BVP

\[ -(\Delta U = 0 \text{, in } \Omega \hspace{1cm} U = Q(g_1), \text{ at } \Gamma_1 \hspace{1cm} U_\nu = g_2, \text{ at } \Gamma_1 \]

admits a solution \( U \) for the Cauchy data \((Q(g_1), g_2)\) and, in this case, \( U = Q(u) \). Since \( Q \) is continuous and invertible, it is enough to prove that \((CP)\) is solvable for a dense subset of \( H^{1/2}(\Gamma_1) \times H^{1/2}_{00}(\Gamma_1)' \). Moreover, due to the superposition principle, it is enough to prove that:

i) For a fixed \( g_1 \in H^{1/2}(\Gamma_1) \), the set of data \( g_2 \) for which \((CP)\) admits a solution is dense in \( [H^{1/2}_{00}(\Gamma_1)]' \);

ii) For a fixed \( g_2 \in [H^{1/2}_{00}(\Gamma_1)]' \), the set of data \( g_1 \) for which \((CP)\) admits a solution is dense in \( H^{1/2}(\Gamma_1) \).

We prove only first item, the proof of the second one being analogous. Let us assume (without loss of generality) that \( g_1 = 0 \). We define

\[ \mathcal{M} := \{ g_2 \in [H^{1/2}_{00}(\Gamma_1)]'; (0, g_2) \text{ is consistent for } (CP) \} \]

If \( \mathcal{M} \) were not dense in \( [H^{1/2}_{00}(\Gamma_1)]' \), the Hahn-Banach’s theorem would guarantee the existence a nonzero continuous linear functional \( \Lambda \in H^{1/2}_{00}(\Gamma_1) \) such that \( \langle \Lambda, g_2 \rangle = 0 \) for all \( g_2 \in \mathcal{M} \). Therefore, the mixed BVP

\[ -\Delta v = 0 \text{, in } \Omega \hspace{1cm} v = \Lambda, \text{ at } \Gamma_1 \hspace{1cm} v_\nu = 0, \text{ at } \Gamma_2, \]

has a unique solution \( v \in H^1(\Omega) \). Likewise, given an arbitrary test function \( \vartheta \in C_0^\infty(\Gamma_2) \), the mixed BVP

\[ -\Delta w = 0 \text{, in } \Omega \hspace{1cm} w = 0, \text{ at } \Gamma_1 \hspace{1cm} w_\nu = \vartheta, \text{ at } \Gamma_2, \]

has a unique solution \( w \in H^1(\Omega) \). Since \( w_\nu|_{\Gamma_1} \in \mathcal{M} \), it follows from integration by parts

\[ 0 = \int_\Omega \Delta w v - \int_\Omega w \Delta v = \int_{\Gamma_1 \cup \Gamma_2} w_\nu v + \int_{\Gamma_1} \Lambda w_\nu + \int_{\Gamma_2} \vartheta v = \int_{\Gamma_2} \vartheta v. \]

Since \( \vartheta \in C_0^\infty(\Gamma_2) \) is arbitrary, \( v|_{\Gamma_2} = 0 \) follows. Therefore, \( v|_{\Gamma_2} = v_\nu|_{\Gamma_2} = 0 \) and \(-\Delta v = 0 \) in \( \Omega \). From the uniqueness of solutions of (linear) Cauchy problems we conclude that \( v = 0 \) in \( \Omega \), contradicting the choice of \( \Lambda \). \( \square \)

---

4Here \( \langle \cdot, \cdot \rangle \) denotes the canonical duality paring between \( H^{1/2}_{00}(\Gamma_1) \) and \( [H^{1/2}_{00}(\Gamma_1)]' \).
References


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Figure 2: First numerical experiment: 0.5-level-sets of the iterates $H_\epsilon(\phi_k)$ for $k=0,1,2,4,6,8,10,13$ (top-left to bottom-right).
Figure 3: First numerical experiment: Level-set function $\phi_k$ for $k=0,4,8,13$. 
Figure 4: First numerical experiment: Results obtained after 100 iterative steps for $a = 0.1, \pi^{-1}, 0.5, 1.0$ (top-left to bottom-right).

Figure 5: First numerical experiment: Results obtained with the iterated Tikhonov-Morosov method after 100 iterations.
Figure 6: Second numerical experiment: Top pictures show reconstruction results for 1% random white noise and bottom pictures for 5% noise.