Introduction to Domain Decomposition Methods

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MOTIVATION

- DD can be used in the framework of any discretization method for PDEs (FEM, FV, FD, SEM) to make their algebraic solution more efficient on parallel computer platforms.

- DDM allow the reformulation of a boundary-value problem on a partition of the computational domain into subdomains ⇒ very convenient framework for the solution of heterogeneous or multiphysics problems, i.e. those that are governed by differential equations of different kinds in different subregions of the computational domain.
THE IDEA

The computational domain $\Omega$ where the bvp is set is subdivided into two or more subdomains in which problems of smaller dimension are to be solved.

Parallel solution algorithms may be used.

There are two ways of subdividing the computational domain:

- with *disjoint* subdomains
- with *overlapping* subdomains

Correspondingly, different DD algorithms will be set up.
AN HISTORICAL REMARK

- The classical “logo” of DDM is [http://www.ddm.org/]

Why this symbol? What does it mean?

- We have to go back to a work of 1869 by Hermann Schwarz...
...who wanted to establish the existence of harmonic functions with prescribed boundary values on regions with non-smooth boundaries:

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial \Omega \end{cases}$$

where

Schwarz's idea: split $\Omega$ into two elementary subdomains:

![Diagram of Ω split into Ω₁ and Ω₂](image)

and set up a suitable iterative method passing information between the subproblems.
MODERN EXAMPLES OF SUBDIVISIONS
REFERENCES

  *Domain Decomposition* 

- A. Quarteroni and A. Valli (1999) 
  *Domain Decomposition Methods for Partial Differential Equations* 

  *Domain Decomposition Methods – Algorithms and Theory* 
  Springer-Verlag, Berlin and Heidelberg.
CLASSICAL ITERATIVE DD METHODS
MODEL PROBLEM

Consider the model problem:

\[
\text{find } u : \Omega \rightarrow \mathbb{R} \text{ s.t. } \begin{cases} \quad Lu = f \quad \text{in } \Omega \\ \quad u = 0 \quad \text{on } \partial \Omega \end{cases}
\]

\(L\) is a generic second order elliptic operator.

The weak formulation reads:

\[
\text{find } u \in V = H^1_0(\Omega) : \quad a(u, v) = (f, v) \quad \forall v \in V
\]

where \(a(\cdot, \cdot)\) is the bilinear form associated with \(L\).
SCHWARZ METHODS

Consider a decomposition of \( \Omega \) with overlap:

\[
\begin{array}{c}
\Omega \\
\Omega_1 \quad \Gamma_2 \\
\Omega_2
\end{array}
\]

Given \( u_2^{(0)} \) on \( \Gamma_1 \), for \( k \geq 1 \):

solve
\[
\begin{cases}
Lu_1^{(k)} = f & \text{in } \Omega_1 \\
u_1^{(k)} = u_2^{(k-1)} & \text{on } \Gamma_1 \\
u_1^{(k)} = 0 & \text{on } \partial \Omega_1 \setminus \Gamma_1
\end{cases}
\]

solve
\[
\begin{cases}
Lu_2^{(k)} = f & \text{in } \Omega_2 \\
u_2^{(k)} = \begin{cases} u_1^{(k)} & \text{on } \Gamma_2 \\
u_1^{(k-1)} & \text{on } \Gamma_2 \end{cases} & \text{on } \partial \Omega_2 \setminus \Gamma_2. \\
u_2^{(k)} = 0 & \text{on } \partial \Omega_2 \setminus \Gamma_2.
\end{cases}
\]
Choice of the trace on $\Gamma_2$:
- if $u_1^{(k)}$ ⇒ multiplicative Schwarz method
- if $u_1^{(k-1)}$ ⇒ additive Schwarz method

We have two elliptic bvp with Dirichlet conditions in $\Omega_1$ and $\Omega_2$, and we wish the two sequences $\{u_1^{(k)}\}$ and $\{u_2^{(k)}\}$ to converge to the restrictions of the solution $u$ of the original problem:

$$\lim_{k \to \infty} u_1^{(k)} = u_{|\Omega_1} \text{ and } \lim_{k \to \infty} u_2^{(k)} = u_{|\Omega_2}.$$

The Schwarz method applied to the model problem always converges, with a rate that increases as the measure $|\Gamma_{12}|$ of the overlapping region $\Gamma_{12}$ increases.
Example
Consider the model problem

\[
\begin{cases}
-u''(x) = 0 & a < x < b, \\
u(a) = u(b) = 0,
\end{cases}
\]

where

\[
\begin{array}{c}
\gamma_2 \\
\gamma_1
\end{array}
\]

The solution is \( u = 0 \). We show a few iterations of the method:

Clearly, the method converges with a rate that reduces as the length of the interval \((\gamma_2, \gamma_1)\) gets smaller.
Remark

- The Schwarz method requires at each iteration the solution of two subproblems of the same kind as those of the original problem.

- The boundary conditions of the initial problem remain unchanged.

- Dirichlet-type boundary conditions are imposed across the interfaces.
NONOVERLAPPING DECOMPOSITION

We partition now the domain $\Omega$ in two disjoint subdomains:

\[ \Omega \quad \Omega_1 \quad \Omega_2 \]

\[ \Gamma \]

The following equivalence result holds.

**Theorem**

The solution $u$ of the model problem is such that $u_{\Omega_i} = u_i$ for $i = 1, 2$, where $u_i$ is the solution to the problem

\[
\begin{cases}
Lu_i = f & \text{in } \Omega_i \\
u_i = 0 & \text{on } \partial \Omega_i \setminus \Gamma
\end{cases}
\]

with interface conditions

\[
u_1 = u_2 \quad \text{and} \quad \frac{\partial u_1}{\partial n_L} = \frac{\partial u_2}{\partial n_L} \quad \text{on } \Gamma
\]

$\partial / \partial n_L$ is the conormal derivative.
Example

The problem

\[
\begin{aligned}
-\text{div} \left( \mu \nabla u \right) &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

is equivalent to

\[
\begin{aligned}
-\text{div} \left( \mu_i \nabla u_i \right) &= f_i \quad \text{in } \Omega_i \\
u &= 0 \quad \text{on } \partial \Omega_i \\
u_1 &= \nu_2 \quad \text{on } \Gamma \\
\mu_1 \frac{\partial \nu_1}{\partial n} &= \mu_2 \frac{\partial \nu_2}{\partial n} \quad \text{on } \Gamma
\end{aligned}
\]

Remark

The proof of the theorem can be done using the weak formulation of the problem. We refer to Lemma 1.2.1 in [QV99].
DIRICHLET-NEUMANN METHOD

Given $u_2^{(0)}$ on $\Gamma$, for $k \geq 1$ solve the problems:

\[
\begin{align*}
L u_1^{(k)} &= f \quad \text{in } \Omega_1 \\
 u_1^{(k)} &= u_2^{(k-1)} \quad \text{on } \Gamma \\
 u_1^{(k)} &= 0 \quad \text{on } \partial \Omega_1 \setminus \Gamma
\end{align*}
\]

\[
\begin{align*}
L u_2^{(k)} &= f \quad \text{in } \Omega_2 \\
 \frac{\partial u_2^{(k)}}{\partial n_L} &= \frac{\partial u_1^{(k)}}{\partial n_L} \quad \text{on } \Gamma \\
 u_2^{(k)} &= 0 \quad \text{on } \partial \Omega_2 \setminus \Gamma
\end{align*}
\]

- The equivalence theorem guarantees that when the two sequences \( \{u_1^{(k)}\} \) and \( \{u_2^{(k)}\} \) converge, their limit will be necessarily the solution to the exact problem.
- The DN algorithm is therefore consistent.
- However, its convergence is not always guaranteed.
Example
Let $\Omega = (a, b)$, $\gamma \in (a, b)$, $L = -d^2/dx^2$ and $f = 0$. At every $k \geq 1$ the DN algorithm generates the two subproblems:

$$
\begin{cases}
-(u_1^{(k)})'' = 0 & a < x < \gamma \\
u_1^{(k)} = u_2^{(k-1)} & x = \gamma \\
u_1^{(k)} = 0 & x = a \\
\end{cases}
$$

$$
\begin{cases}
-(u_2^{(k)})'' = 0 & \gamma < x < b \\
(u_2^{(k)})' = (u_1^{(k)})' & x = \gamma \\
u_2^{(k)} = 0 & x = b.
\end{cases}
$$

The two sequences converge only if $\gamma > (a + b)/2$:
A variant of the DN algorithm can be set up by replacing the Dirichlet condition in the first subdomain by

$$u_1^{(k)} = \theta u_2^{(k-1)} + (1 - \theta) u_1^{(k-1)} \quad \text{on } \Gamma$$

using a *relaxation* parameter $\theta > 0$.

In such a way it is always possible to reduce the error between two subsequent iterates.

In the previous example, we can easily verify that, by choosing

$$\theta_{opt} = -\frac{u_1^{(k-1)}}{u_2^{(k-1)} - u_1^{(k-1)}}$$

the algorithm converges to the exact solution in a single iteration.

More in general, there exists a suitable value $0 < \theta_{max} < 1$ such that the DN algorithm converges for any possible choice of the relaxation parameter $\theta$ in the interval $(0, \theta_{max})$. 
NEUMANN-NEUMANN ALGORITHM

Consider again a partition of $\Omega$ into two disjoint subdomains and denote by $\lambda$ the (unknown) value of the solution $u$ on their interface $\Gamma$:

$$\lambda = u_i \text{ on } \Gamma \quad (i = 1, 2)$$

Consider the following iterative algorithm: for any given $\lambda^{(0)}$ on $\Gamma$, for $k \geq 0$ and $i = 1, 2$, solve the following problems:

$$\left\{ \begin{array}{ll}
L u_i^{(k+1)} = f & \text{in } \Omega_i \\
u_i^{(k+1)} = \lambda^{(k)} & \text{on } \Gamma \\
u_i^{(k+1)} = 0 & \text{on } \partial \Omega_i \setminus \Gamma
\end{array} \right.$$  

$$\left\{ \begin{array}{ll}
L \psi_i^{(k+1)} = 0 & \text{in } \Omega_i \\
\frac{\partial \psi_i^{(k+1)}}{\partial n} = \frac{\partial u_1^{(k+1)}}{\partial n} - \frac{\partial u_2^{(k+1)}}{\partial n} & \text{on } \Gamma \\
\psi_i^{(k+1)} = 0 & \text{on } \partial \Omega_i \setminus \Gamma
\end{array} \right.$$  

with

$$\lambda^{(k+1)} = \lambda^{(k)} - \theta \left( \sigma_1 \psi_1^{(k+1)}|_{\Gamma} - \sigma_2 \psi_2^{(k+1)}|_{\Gamma} \right)$$

where $\theta$ is a positive acceleration parameter, while $\sigma_1$ and $\sigma_2$ are two positive coefficients.
ROBIN-ROBIN ALGORITHM

For every $k \geq 0$ solve the following problems:

\[
\begin{align*}
\begin{cases}
  u_1^{(k+1)} = f & \text{in } \Omega_1 \\
n_1^{(k+1)} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma \\
\frac{\partial u_1^{(k+1)}}{\partial n} + \gamma_1 u_1^{(k+1)} = \frac{\partial u_2^{(k)}}{\partial n} + \gamma_1 u_2^{(k)} & \text{on } \Gamma
\end{cases}
\end{align*}
\]

then

\[
\begin{align*}
\begin{cases}
  Lu_2^{(k+1)} = f & \text{in } \Omega_2 \\
n_2^{(k+1)} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma \\
\frac{\partial u_2^{(k+1)}}{\partial n} + \gamma_2 u_2^{(k+1)} = \frac{\partial u_1^{(k+1)}}{\partial n} + \gamma_2 u_1^{(k+1)} & \text{on } \Gamma
\end{cases}
\end{align*}
\]

where $u_2^{(0)}$ is assigned and $\gamma_1, \gamma_2$ are non-negative acceleration parameters that satisfy $\gamma_1 + \gamma_2 > 0$.

Aiming at parallelization, we could use $u_1^{(k)}$ instead of $u_1^{(k+1)}$. 
THE STEKLOV-POINCARÉ INTERFACE EQUATION
MULTI-DOMAIN FORMULATION OF POISSON PROBLEM AND INTERFACE CONDITIONS

We consider now the model problem:

\[
\begin{align*}
-\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

For a domain partitioned into two disjoint subdomains, we can write the equivalent multi-domain formulation \((u_i = u|_{\Omega_i}, \ i = 1, 2)\):

\[
\begin{align*}
-\Delta u_1 &= f \quad \text{in } \Omega_1 \\
u_1 &= 0 \quad \text{on } \partial\Omega_1 \setminus \Gamma \\
-\Delta u_2 &= f \quad \text{in } \Omega_2 \\
u_2 &= 0 \quad \text{on } \partial\Omega_2 \setminus \Gamma \\
u_1 &= u_2 \quad \text{on } \Gamma \\
\frac{\partial u_1}{\partial n} &= \frac{\partial u_2}{\partial n} \quad \text{on } \Gamma.
\end{align*}
\]
Remark

- On the interface $\Gamma$ we have the normal unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$:

- There holds: $\mathbf{n}_1 = -\mathbf{n}_2$ on $\Gamma$.

- We denote $\mathbf{n} = \mathbf{n}_1$ so that

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial n_1} = -\frac{\partial}{\partial n_2}$$

on $\Gamma$. 
Let $\lambda$ be the unknown value of the solution $u$ on the interface $\Gamma$:

$$\lambda = u|_{\Gamma}$$

Should we know a priori the value $\lambda$ on $\Gamma$, we could solve the following two independent boundary-value problems with Dirichlet condition on $\Gamma$ ($i = 1, 2$):

$$\begin{cases} 
-\Delta w_i = f & \text{in } \Omega_i \\
 w_i = 0 & \text{on } \partial\Omega_i \setminus \Gamma \\
w_i = \lambda & \text{on } \Gamma.
\end{cases}$$
With the aim of obtaining the value $\lambda$ on $\Gamma$, let us split $w_i$ as follows

$$w_i = w_i^* + u_i^0,$$

where $w_i^*$ and $u_i^0$ represent the solutions of the following problems ($i = 1, 2$):

$$
\begin{align*}
-\Delta w_i^* &= f \quad \text{in } \Omega_i \\
 w_i^* &= 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega \\
 w_i^* &= 0 \quad \text{on } \Gamma
\end{align*}
$$

and

$$
\begin{align*}
-\Delta u_i^0 &= 0 \quad \text{in } \Omega_i \\
 u_i^0 &= 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega \\
 u_i^0 &= \lambda \quad \text{on } \Gamma.
\end{align*}
$$
• The functions \( w_i^* \) depend solely on the source data \( f \)

\[ \Rightarrow \quad w_i^* = G_i f \]

where \( G_i \) is a linear continuous operator

• \( u_i^0 \) depend solely on the value \( \lambda \) on \( \Gamma \)

\[ \Rightarrow \quad u_i^0 = H_i \lambda \]

where \( H_i \) is the so-called harmonic extension operator of \( \lambda \) on the domain \( \Omega_i \).

We have that

\[ u_i = w_i^* + u_i^0 \quad (i = 1, 2) \]

\[ \Leftrightarrow \quad \frac{\partial w_1}{\partial n} = \frac{\partial w_2}{\partial n} \quad \text{on } \Gamma \]

\[ \Leftrightarrow \quad \frac{\partial}{\partial n}(w_1^* + u_1^0) = \frac{\partial}{\partial n}(w_2^* + u_2^0) \quad \text{on } \Gamma \]

\[ \Leftrightarrow \quad \frac{\partial}{\partial n}(G_1 f + H_1 \lambda) = \frac{\partial}{\partial n}(G_2 f + H_2 \lambda) \quad \text{on } \Gamma \]

\[ \Leftrightarrow \quad \left( \frac{\partial H_1}{\partial n} - \frac{\partial H_2}{\partial n} \right) \lambda = \left( \frac{\partial G_2}{\partial n} - \frac{\partial G_1}{\partial n} \right) f \quad \text{on } \Gamma. \]
We have obtained the *Steklov-Poincaré equation* for the unknown $\lambda$ on the interface $\Gamma$:

$$S\lambda = \chi \quad \text{on} \quad \Gamma$$

- $S$ is the *Steklov-Poincaré pseudo-differential operator*:

$$S\mu = \frac{\partial}{\partial n} H_1 \mu - \frac{\partial}{\partial n} H_2 \mu = \sum_{i=1}^{2} \frac{\partial}{\partial n_i} H_i \mu$$

- $\chi$ is a linear functional which depends on $f$:

$$\chi = \frac{\partial}{\partial n} G_2 f - \frac{\partial}{\partial n} G_1 f = - \sum_{i=1}^{2} \frac{\partial}{\partial n_i} G_i f.$$ 

- The operator

$$S_i : \mu \rightarrow S_i \mu = \frac{\partial}{\partial n_i} (H_i \mu) \bigg|_{\Gamma} \quad i = 1, 2,$$

is called *local Steklov-Poincaré operator* (*Dirichlet-to-Neumann*) which operates between the trace space

$$\Lambda = \{ \mu : \exists v \in V \text{ s.t. } \mu = v|_{\Gamma} \} = H^{1/2}_{00}(\Gamma)$$

and its dual $\Lambda'$.
Example

To provide an example of the operator $S$, we consider a simple 1D problem.
Let $\Omega = (a, b) \subset \mathbb{R}$ as illustrated below.
We split $\Omega$ in two nonoverlapping subdomains. In this case the interface $\Gamma$ reduces to the point $\gamma \in (a, b)$ and the Steklov-Poincaré operator $S$ becomes

$$S\lambda = \left( \frac{dH_1}{dx} - \frac{dH_2}{dx} \right) \lambda = \left( \frac{1}{l_1} + \frac{1}{l_2} \right) \lambda$$

where $l_1 = \gamma - a$ et $l_2 = b - \gamma$. 

![Diagram](attachment://example_diagram.png)
EQUIVALENCE BETWEEN THE DD SCHEMES AND CLASSICAL ITERATIVE METHODS
The DN method can be reinterpreted as a preconditioned Richardson method for the solution of the Steklov-Poincaré interface equation:

\[ P_{DN}(\lambda^{(k)} - \lambda^{(k-1)}) = \theta(\chi - S\lambda^{(k-1)}) \]

The preconditioning operator is \( P_{DN} = S_2 = \partial(H_2\mu) / \partial n_2 \).

The NN method can also be interpreted as a preconditioned Richardson algorithm

\[ P_{NN}(\lambda^{(k)} - \lambda^{(k-1)}) = \theta(\chi - S\lambda^{(k-1)}) \]

with \( P_{NN} = (\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1} \).
The Robin-Robin algorithm is equivalent to the following alternating direction (ADI) algorithm:

\[
(\gamma_1 I_{\Lambda} + S_1)\mu_1^{(k)} = \chi + (\gamma_1 I_{\Lambda} + S_2)\mu_2^{(k-1)},
\]
\[
(\gamma_2 I_{\Lambda} + S_2)\mu_2^{(k)} = \chi + (\gamma_2 I_{\Lambda} + S_1)\mu_1^{(k-1)},
\]

where \(I_{\Lambda} : \Lambda \to \Lambda'\) here denotes the Riesz isomorphism between the Hilbert space \(\Lambda\) and its dual \(\Lambda'\).

At convergence (for a convenient choice of \(\gamma_1\) and \(\gamma_2\)), we have \(\mu_1 = \mu_2 = \lambda\).
FINITE ELEMENT APPROXIMATION:
MULTI-DOMAIN FORMULATION
Consider the Poisson problem:

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

Its weak formulation reads

\[
\text{find } u \in V : \quad a(u, v) = F(v) \quad \forall v \in V
\]

where \( V = H^1_0(\Omega) \),

\[
a(v, w) = \int_{\Omega} \nabla v \cdot \nabla w \quad \forall v, w \in V
\]

and

\[
F(v) = \int_{\Omega} f v \quad \forall v \in V.
\]
Suppose that $\Omega$ is split into two nonoverlapping subdomains and consider a uniform triangulation $\mathcal{T}_h$ of $\Omega$, *conforming* on $\Gamma$: 

\[ \Omega_1 \]

\[ \Gamma \]

\[ \Omega_2 \]
The Galerkin finite element approximation of the Poisson problem reads:

\[
\text{find } u_h \in V_h : \quad a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h
\]  

(1)

where

\[ V_h = \{ v_h \in C^0(\overline{\Omega}) : v_h|_K \in \mathbb{P}_r \quad r \geq 1, \quad \forall K \in T_h, \quad v_h = 0 \text{ on } \partial \Omega \} \]

is the space of finite element functions of degree \( r \) with basis \( \{\varphi_i\}_{i=1}^{N_h} \).

The Galerkin approximation (1) is equivalent to:

\[
\text{find } u_h \in V_h : \quad a(u_h, \varphi_i) = F(\varphi_i) \quad \forall i = 1, \ldots, N_h.
\]  

(2)
We partition the nodes of the triangulation as follows:

- \( \{x_j^{(1)}, \ 1 \leq j \leq N_1\} \) nodes in subdomain \( \Omega_1 \)
- \( \{x_j^{(2)}, \ 1 \leq j \leq N_2\} \) nodes in subdomain \( \Omega_2 \)
- \( \{x_j^{(\Gamma)}, \ 1 \leq j \leq N_\Gamma\} \) nodes on the interface \( \Gamma \),

and we split the basis functions accordingly:

- \( \varphi_j^{(1)} \) functions associated to the nodes \( x_j^{(1)} \)
- \( \varphi_j^{(2)} \) functions associated to the nodes \( x_j^{(2)} \)
- \( \varphi_j^{(\Gamma)} \) functions associated to the nodes \( x_j^{(\Gamma)} \) on the interface.
Example
Problem (2) can be equivalently rewritten as:

\[
\text{find } u_h \in V_h : \begin{cases} 
    a(u_h, \varphi_i^{(1)}) = F(\varphi_i^{(1)}) & \forall i = 1, \ldots, N_1 \\
    a(u_h, \varphi_j^{(2)}) = F(\varphi_j^{(2)}) & \forall j = 1, \ldots, N_2 \\
    a(u_h, \varphi_k^{(\Gamma)}) = F(\varphi_k^{(\Gamma)}) & \forall k = 1, \ldots, N_{\Gamma}.
\end{cases}
\] (3)

We introduce the bilinear form on \( \Omega_i \):

\[
a_i(v, w) = \int_{\Omega_i} \nabla v \cdot \nabla w \\n\forall v, w \in V, \; i = 1, 2,
\]

the space

\[V^i_h = \{ v \in H^1(\Omega_i) : v = 0 \text{ on } \partial \Omega_i \setminus \Gamma \} \quad (i = 1, 2)\]

and let \( u_h^{(i)} = u_h|_{\Omega_i} \) \( (i = 1, 2) \).

Then, problem (3) can be written equivalently as:

find \( u_h^{(1)} \in V^1_h, u_h^{(2)} \in V^2_h \) s.t.

\[
\begin{cases} 
    a_1(u_h^{(1)}, \varphi_i^{(1)}) = F_1(\varphi_i^{(1)}) & \forall i = 1, \ldots, N_1 \\
    a_2(u_h^{(2)}, \varphi_j^{(2)}) = F_2(\varphi_j^{(2)}) & \forall j = 1, \ldots, N_2 \\
    a_1(u_h^{(1)}, \varphi_k^{(\Gamma)}|_{\Omega_1}) + a_2(u_h^{(2)}, \varphi_k^{(\Gamma)}|_{\Omega_2}) \\
    = F_1(\varphi_k^{(\Gamma)}|_{\Omega_1}) + F_2(\varphi_k^{(\Gamma)}|_{\Omega_2}) & \forall k = 1, \ldots, N_{\Gamma}.
\end{cases}
\] (4)
Remark

- Problem (4) corresponds to the finite element approximation of the multi-domain formulation of the Poisson problem:

\[
\begin{aligned}
-\triangle u_1 &= f & \text{in } \Omega_1 \\
u_1 &= 0 & \text{on } \partial\Omega_1 \setminus \Gamma \\
-\triangle u_2 &= f & \text{in } \Omega_2 \\
u_2 &= 0 & \text{on } \partial\Omega_2 \setminus \Gamma \\
u_1 &= u_2 & \text{on } \Gamma \\
\frac{\partial u_1}{\partial n} &= \frac{\partial u_2}{\partial n} & \text{on } \Gamma.
\end{aligned}
\]

- The condition $u_1 = u_2$ on $\Gamma$ is satisfied by definition of $u_h^{(i)}$. 
We can write:

\[ u_h(x) = \sum_{j=1}^{N_1} u_h(x_j^{(1)}) \varphi_j^{(1)}(x) + \sum_{j=1}^{N_2} u_h(x_j^{(2)}) \varphi_j^{(2)}(x) \]

\[ + \sum_{j=1}^{N_\Gamma} u_h(x_j^{(\Gamma)}) \varphi_j^{(\Gamma)}(x), \]

and we substitute this expression in (4) to obtain...
\[
\begin{array}{l}
\sum_{j=1}^{N_1} u_h(x_{j}^{(1)}) a_1(\varphi_{j}^{(1)}, \varphi_{i}^{(1)}) + \sum_{j=1}^{N_r} u_h(x_{j}^{(\Gamma)}) a_1(\varphi_{j}^{(\Gamma)}, \varphi_{i}^{(1)}) = F_1(\varphi_{i}^{(1)}) \quad \forall i = 1, \ldots, N_1 \\
\sum_{j=1}^{N_2} u_h(x_{j}^{(2)}) a_2(\varphi_{j}^{(2)}, \varphi_{i}^{(2)}) + \sum_{j=1}^{N_r} u_h(x_{j}^{(\Gamma)}) a_2(\varphi_{j}^{(\Gamma)}, \varphi_{i}^{(2)}) = F_2(\varphi_{i}^{(2)}) \quad \forall i = 1, \ldots, N_2 \\
\sum_{j=1}^{N_r} u_h(x_{j}^{(\Gamma)}) \left[ a_1(\varphi_{j}^{(\Gamma)}, \varphi_{i}^{(\Gamma)}) + a_2(\varphi_{j}^{(\Gamma)}, \varphi_{i}^{(\Gamma)}) \right] \\
+ \sum_{j=1}^{N_1} u_h(x_{j}^{(1)}) a_1(\varphi_{j}^{(1)}, \varphi_{i}^{(\Gamma)}) + \sum_{j=1}^{N_2} u_h(x_{j}^{(2)}) a_2(\varphi_{j}^{(2)}, \varphi_{i}^{(\Gamma)}) \\
= F_1(\varphi_{i}^{(\Gamma)}|\Omega_1) + F_2(\varphi_{i}^{(\Gamma)}|\Omega_2) \quad \forall i = 1, \ldots, N_\Gamma.
\end{array}
\]
A bit of algebra...

\[
\begin{aligned}
\sum_{j=1}^{N_1} u_h(x^{(1)}_j)(A_{11})_{ij} + \sum_{j=1}^{N_\Gamma} u_h(x^{(\Gamma)}_j)(A_{1\Gamma})_{ij} &= F_1(\varphi^{(1)}_i) \quad \forall i = 1, \ldots, N_1 \\
\sum_{j=1}^{N_2} u_h(x^{(2)}_j)(A_{22})_{ij} + \sum_{j=1}^{N_\Gamma} u_h(x^{(\Gamma)}_j)(A_{2\Gamma})_{ij} &= F_2(\varphi^{(2)}_i) \quad \forall i = 1, \ldots, N_2 \\
\sum_{j=1}^{N_\Gamma} u_h(x^{(\Gamma)}_j) \left[ (A^{(1)}_{\Gamma\Gamma})_{ij} + (A^{(2)}_{\Gamma\Gamma})_{ij} \right] \\
+ \sum_{j=1}^{N_1} u_h(x^{(1)}_j)(A_{\Gamma1})_{ij} + \sum_{j=1}^{N_2} u_h(x^{(2)}_j)(A_{\Gamma2})_{ij} \\
= F_1(\varphi^{(\Gamma)}_i|_{\Omega_1}) + F_2(\varphi^{(\Gamma)}_i|_{\Omega_2}) \quad \forall i = 1, \ldots, N_\Gamma.
\end{aligned}
\]
A bit of algebra... [continued]

\[
\begin{align*}
\sum_{j=1}^{N_1} u_1 (A_{11})_{ij} + \sum_{j=1}^{N_\Gamma} u_{\Gamma} (A_{1\Gamma})_{ij} &= f_1 & \forall i = 1, \ldots, N_1 \\
\sum_{j=1}^{N_2} u_2 (A_{22})_{ij} + \sum_{j=1}^{N_\Gamma} u_{\Gamma} (A_{2\Gamma})_{ij} &= f_2 & \forall i = 1, \ldots, N_2 \\
\sum_{j=1}^{N_\Gamma} u_{\Gamma} \left[ (A_{\Gamma\Gamma}^{(1)})_{ij} + (A_{\Gamma\Gamma}^{(2)})_{ij} \right] \\
+ \sum_{j=1}^{N_1} u_1 (A_{\Gamma1})_{ij} + \sum_{j=1}^{N_2} u_2 (A_{\Gamma2})_{ij} &= f_1^{(\Gamma)} + f_2^{(\Gamma)} & \forall i = 1, \ldots, N_\Gamma.
\end{align*}
\]
... so that we obtain the algebraic form:

\[
\begin{align*}
A_{11}u_1 + A_{1\Gamma} \lambda &= f_1 \\
A_{22}u_2 + A_{2\Gamma} \lambda &= f_2 \\
A_{\Gamma 1}u_1 + A_{\Gamma 2}u_2 + (A_{\Gamma \Gamma}^{(1)} + A_{\Gamma \Gamma}^{(2)}) \lambda &= f_1^{(\Gamma)} + f_2^{(\Gamma)}
\end{align*}
\]

or

\[
\begin{pmatrix}
A_{11} & 0 & A_{1\Gamma} \\
0 & A_{22} & A_{2\Gamma} \\
A_{\Gamma 1} & A_{\Gamma 2} & A_{\Gamma \Gamma}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\lambda
\end{pmatrix}
= 
\begin{pmatrix}
f_1 \\
f_2 \\
f_{\Gamma}
\end{pmatrix}
\]

where we denoted \( A_{\Gamma \Gamma} = (A_{\Gamma \Gamma}^{(1)} + A_{\Gamma \Gamma}^{(2)}) \) and \( f_{\Gamma} = f_1^{(\Gamma)} + f_2^{(\Gamma)} \).
THE SCHUR COMPLEMENT SYSTEM
Since $\lambda$ represents the unknown value of $u$ on $\Gamma$, its finite element correspondent is the vector $\lambda$ of the values of $u_h$ at the interface nodes.

By Gaussian elimination, we can obtain a new reduced system in the sole unknown $\lambda$:

- Matrices $A_{11}$ and $A_{22}$ are invertible since they are associated with two homogeneous Dirichlet boundary-value problems for the Laplace operator:

$$u_1 = A_{11}^{-1} (f_1 - A_{1\Gamma} \lambda) \quad \text{and} \quad u_2 = A_{22}^{-1} (f_2 - A_{2\Gamma} \lambda). \quad (7)$$

- From the third equation we obtain:

$$\left[ \left( A_{1\Gamma}^{(1)} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} \right) + \left( A_{2\Gamma}^{(2)} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma} \right) \right] \lambda = f_{\Gamma} - A_{\Gamma 1} A_{11}^{-1} f_1 - A_{\Gamma 2} A_{22}^{-1} f_2.$$
Setting

\[ \Sigma = \Sigma_1 + \Sigma_2 \quad \text{with} \quad \Sigma_i = A_{i\Gamma}^{(i)} - A_{i\Gamma} A_{ii}^{-1} A_{i\Gamma} \quad (i = 1, 2) \]

and

\[ \chi_{\Gamma} = f_{\Gamma} - A_{\Gamma 1} A_{11}^{-1} f_1 - A_{\Gamma 2} A_{22}^{-1} f_2 \]

we obtain the \textit{Schur complement system}

\[ \Sigma \lambda = \chi_{\Gamma} \]

- \( \Sigma \) and \( \chi_{\Gamma} \) approximate \( S \) and \( \chi \).

- \( \Sigma \) is the so-called \textit{Schur complement} of \( A \) with respect to \( u_1 \) and \( u_2 \).

- \( \Sigma_i \) are the Schur complements related to the subdomains \( \Omega_i \) \((i = 1, 2)\).
Remark

- After solving the Schur complement system in $\lambda$, thanks to (7) we can compute $u_1$ and $u_2$.

- This is equivalent to solving two Poisson problems in the subdomains $\Omega_1$ and $\Omega_2$ with the Dirichlet boundary condition $u^{(i)}_h |_{\Gamma} = \lambda_h$ ($i = 1, 2$) on the interface $\Gamma$. 
PROPERTIES OF THE SCHUR COMPLEMENT $\Sigma$

The Schur complement $\Sigma$ inherits some of the properties of $A$:

- if $A$ is singular, so is $\Sigma$;
- if $A$ (respectively, $A_{ii}$) is symmetric, then $\Sigma$ (respectively, $\Sigma_i$) is symmetric too;
- if $A$ is positive definite, so is $\Sigma$.

Moreover, concerning the condition number, we have

- $\kappa(A) \approx Ch^{-2}$
- $\kappa(\Sigma) \approx Ch^{-1}$
PRECONDITIONERS FOR THE SCHUR COMPLEMENT SYSTEM
The iterative methods that we have illustrated are equivalent to preconditioned Richardson methods for the Schur complement system with preconditioners:

- for the DN algorithm: $P_h = \Sigma_2$

- for the ND algorithm: $P_h = \Sigma_1$

- for the NN algorithm: $P_h = (\sigma_1 \Sigma_1^{-1} + \sigma_2 \Sigma_2^{-1})^{-1}$

- for the RR algorithm: $P_h = (\gamma_1 + \gamma_2)^{-1}(\gamma_1 I + \Sigma_1)(\gamma_2 I + \Sigma_2)$
All these preconditioners are optimal in the sense of the following definition.

**Definition**

A preconditioner $P$ is **optimal** for a matrix $A \in \mathbb{R}^{N \times N}$ if the condition number of $P^{-1}A$ is bounded uniformly with respect to the dimension $N$ of $A$.

In particular, we have

1. $\kappa((\Sigma^{-1})_i \Sigma) = O(1) \quad (i = 1, 2)$

2. $\kappa(((\sigma_1 \Sigma^{-1}_1 + \sigma_2 \Sigma^{-1}_2) \Sigma) = O(1)$

for all $\sigma_1, \sigma_2 > 0$
NONOVERLAPPING METHODS IN THE CASE OF MORE THAN TWO SUBDOMAINS
MULTI-DOMAIN FORMULATION FOR $M > 2$ SUBDOMAINS

We generalize now the nonoverlapping methods to the case of a domain $\Omega$ split into $M > 2$ subdomains:

- $\Omega_i \ (i = 1, \ldots, M)$ such that $\bigcup \overline{\Omega}_i = \overline{\Omega}$
- $\Gamma_i = \partial \Omega_i \setminus \partial \Omega$
- $\Gamma = \bigcup \Gamma_i$. 

\[
\begin{align*}
\Omega & = \bigcup \overline{\Omega}_i \\
\Gamma & = \bigcup \Gamma_i
\end{align*}
\]
At the differential level, we have the equivalent multi-domain formulation:

\[
\begin{align*}
-\triangle u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

\[
\Leftrightarrow
\left\{
\begin{array}{l}
-\triangle u_i = f \quad \text{in } \Omega_i \quad (i = 1, \ldots, M) \\
u_i = u_k \\
\frac{\partial u_i}{\partial n_i} = \frac{\partial u_k}{\partial n_i} \\
u_i = 0
\end{array}ight.
\text{on } \Gamma_{ik}
\]

where \( \Gamma_{ik} = \partial \Omega_i \cap \partial \Omega_k \neq \emptyset \).
FINITE ELEMENT APPROXIMATION

- Considering a conforming finite element approximation, we obtain the linear system:

\[
\begin{pmatrix}
A_{II} & A_{I\Gamma} \\
A_{\Gamma I} & A_{\Gamma\Gamma}
\end{pmatrix}
\begin{pmatrix}
u_I \\
\lambda
\end{pmatrix} =
\begin{pmatrix}
f_I \\
f_\Gamma
\end{pmatrix}
\]

where \(u_I\) is the vector of unknowns in the internal nodes and \(\lambda\) is the vector of unknowns on \(\Gamma\): \(\lambda = u_\Gamma\).

- The submatrix \(A_{II}\) associated to the internal nodes is block-diagonal:

\[
A_{II} =
\begin{pmatrix}
A_{11} & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & A_{MM}
\end{pmatrix}
\]

- \(A_{I\Gamma}\) is a banded matrix (interactions with local interfaces).
**Remark**

On each subdomain $\Omega_i$ the matrix

$$A_i = \begin{pmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma,\Gamma i} \end{pmatrix}$$

represents the local stiffness matrix associated to a Neumann problem.
THE SCHUR COMPLEMENT SYSTEM

Since $A_\parallel$ is non-singular, from (8) we can derive

$$u_\parallel = A_\parallel^{-1}(f_\parallel - A_\parallel f_\Gamma u_\Gamma) \quad (9)$$

By eliminating $u_\parallel$ from (8), we get

$$A_\Gamma \lambda = f_\Gamma - A_\Gamma A_\parallel^{-1}(f_\parallel - A_\parallel \lambda)$$

that is

$$(A_\Gamma - A_\Gamma A_\parallel^{-1}A_\parallel) \lambda = f_\Gamma - A_\Gamma A_\parallel^{-1}f_\parallel. \quad (10)$$

Denoting

$$\Sigma = A_\Gamma - A_\Gamma A_\parallel^{-1}A_\parallel$$

and

$$\chi_\Gamma = f_\Gamma - A_\Gamma A_\parallel^{-1}f_\parallel$$

we obtain the Schur complement system in the multi-domain case

$$\Sigma \lambda = \chi_\Gamma \quad (11)$$
On the left, decomposition of $\Omega = (0, 1) \times (0, 1)$ into four square subdomains. On the right, sparsity pattern of the Schur complement matrix arising from the decomposition depicted on the left.

Remark

The local Schur complements are defined as

$$\Sigma_i = A_{\Gamma i \Gamma i} - A_{\Gamma i \cdot} A_{ii}^{-1} A_{i \Gamma i}$$

so that

$$\Sigma = R_{\Gamma 1}^T \Sigma_1 R_{\Gamma 1} + \ldots + R_{\Gamma M}^T \Sigma_M R_{\Gamma M}$$

where $R_{\Gamma i}$ is the restriction from the skeleton $\Gamma$ to the boundary of $\Omega_i$. 
A SIMPLE ALGORITHM

To compute a FE approximation of the solution $u$ of the Poisson problem

$$\begin{cases}
-\triangle u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

we can do the following steps:

1. solve the Schur complement system

$$\Sigma \lambda = \chi_{\Gamma}$$

   to compute $\lambda$ on the whole interface $\Gamma$;

2. solve

$$A_{II}u_I = f_I - A_{I\Gamma} \lambda$$

   i.e., $M$ independent problems of reduced dimension

$$A_{ii}u_i^i = g^i \quad (i = 1, \ldots, M)$$

   possibly in parallel.
The following estimate can be proved for the condition number of the Schur complement matrix $\Sigma$:

there exists a constant $C > 0$, independent of $h$ and $H$ such that

$$\kappa(\Sigma) \leq C \frac{H_{\text{max}}}{h H_{\text{min}}^2}$$

where $H_{\text{max}}$ and $H_{\text{min}}$ are the maximal and minimal diameters of the subdomains, respectively.
NUMERICAL RESULTS

We consider the model problem

\[
\begin{cases}
-\Delta u &= f \quad \text{in } \Omega = (0, 1) \times (0, 1) \\
   u &= 0 \quad \text{on } \partial\Omega
\end{cases}
\]

We decompose \( \Omega \) into \( M \) square regions \( \Omega_i \) of characteristic dimension \( H \) such that \( \bigcup_{i=1}^M \Omega_i = \Omega \):

(Decomposition of \( \Omega = (0, 1) \times (0, 1) \) into four square subdomains)
Condition number of $\Sigma$:

<table>
<thead>
<tr>
<th>$\kappa(\Sigma)$</th>
<th>$H = 1/2$</th>
<th>$H = 1/4$</th>
<th>$H = 1/8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h=1/8$</td>
<td>9.77</td>
<td>14.83</td>
<td>25.27</td>
</tr>
<tr>
<td>$h=1/16$</td>
<td>21.49</td>
<td>35.25</td>
<td>58.60</td>
</tr>
<tr>
<td>$h=1/32$</td>
<td>44.09</td>
<td>75.10</td>
<td>137.73</td>
</tr>
<tr>
<td>$h=1/64$</td>
<td>91.98</td>
<td>155.19</td>
<td>290.43</td>
</tr>
</tbody>
</table>

Notice the dependence on $\frac{1}{h}$ and $\frac{1}{H}$.
The Schur complement $\Sigma$ is \textit{dense}.

If we use iterative solvers to solve the Schur complement system

$$\Sigma \lambda = \chi_\Gamma$$

it is not convenient from the point of view of memory usage to explicitly compute the elements of $\Sigma$.

We need only to \textit{compute the product $\Sigma x_\Gamma$} for any given vector $x_\Gamma$. 
For example, in the *conjugate gradient method*:

a) Choose a starting solution $\lambda_0$

b) Compute $r_0 = \chi \Gamma - \Sigma \lambda_0$, $p_0 = r_0$

For $k \geq 0$ until convergence, Do

c) $\alpha_k = \frac{(p_k, r_k)}{(\Sigma p_k, p_k)}$

d) $\lambda_{k+1} = \lambda_k + \alpha_k p_k$

e) $r_{k+1} = r_k - \alpha_k \Sigma p_k$

f) $\beta_k = \frac{(\Sigma p_k, r_{k+1})}{(\Sigma p_k, p_k)}$

g) $p_{k+1} = r_{k+1} - \beta_k p_k$

EndFor
SCHUR COMPLEMENT MULTIPLICATION BY A VECTOR

We can define the restriction and prolongation operators $R_{\Gamma_i}$ and $R^{T}_{\Gamma_i}$ that act on the vector of the interface nodal values.

Recalling that

$$\Sigma = \Sigma_1 + \ldots + \Sigma_M \quad \text{with} \quad \Sigma_i = A_{\Gamma_i\Gamma_i} - A_{\Gamma_i;i} A^{-1}_{ii} A_{i\Gamma_i}$$

we can use the following *fully parallel algorithm*.

Given $x_\Gamma$, compute $y_\Gamma = \Sigma x_\Gamma$ as follows:

a) Set $y_\Gamma = 0$

For $i = 1, \ldots, M$ Do in parallel:

b) $x_i = R_{\Gamma_i} x_\Gamma$

c) $z_i = A_{ii\Gamma_i} x_i$

d) $z_i \leftarrow A^{-1}_{ii} z_i$

e) sum to the global vector $y_{\Gamma_i} \leftarrow A_{\Gamma_i;\Gamma_i} x_\Gamma - A_{\Gamma_i; i} z_i$

f) sum to the global vector $y_\Gamma \leftarrow R^{T}_{\Gamma_i} y_{\Gamma_i}$

EndFor
Before using for the first time the Schur complement matrix, an *off-line startup phase* is required:

For $i = 1, \ldots, M$ Do in parallel:

a) Built the matrix $A_i$

b) Reorder $A_i$ as

$$A_i = \begin{pmatrix} A_{ii} & A_{i\Gamma_i} \\ A_{\Gamma_i i} & A_{\Gamma_i \Gamma_i} \end{pmatrix}$$

and extract the submatrices $A_{ii}$, $A_{i\Gamma_i}$, $A_{\Gamma_i i}$ and $A_{\Gamma_i \Gamma_i}$

c) Compute the LU or Cholesky factorization of $A_{ii}$

EndFor
SCALABILITY

**Definition**

A preconditioner \( P_h \) of \( \Sigma \) is said to be **scalable** if the condition number of the preconditioned matrix \( P_h^{-1} \Sigma \) is independent of the number of subdomains.

Iterative methods using scalable preconditioners allow henceforth to achieve convergence rates independent of the subdomain number.

**Remark**

General strategy:

\[
P_h^{-1} = \sum_{i=1}^{M} R_{\Gamma i}^T P_{i,h}^{-1} R_{\Gamma i} + R_{\Gamma}^T P_H^{-1} R_{\Gamma}
\]

where \( P_{i,h} \) is a suitable preconditioner for the local Schur complement \( \Sigma_i \). The last term is the coarse-scale correction.
NEUMANN-NEUMANN PRECONDITIONER

The Neumann-Neumann preconditioner for more subdomains reads:

\[(P_{h}^{NN})^{-1} = \sum_{i=1}^{M} R_{\Gamma i}^T D_i \Sigma_i^* D_i R_{\Gamma i}\]

where \(\Sigma_i^*\) is either \(\Sigma_i^{-1}\) or an approximation of \(\Sigma_i^{-1}\).

\(D_i\) is a diagonal matrix of positive weights

\[D_i = \begin{pmatrix} d_1 & \cdots & \cdots & d_n \end{pmatrix}\]

\(d_j\) is the number of subdomains that share the \(j\)-th node.

We have the following estimate:

\[\kappa ((P_{h}^{NN})^{-1} \Sigma) \leq CH^{-2} \left(1 + \log \frac{H}{h}\right)^2\]
Remark

- The presence of $D_i$ and $R_{\Gamma_i}$ only entails matrix-matrix multiplications.
- Applying $\Sigma_i^{-1}$ may be done using local inverses: if $q$ is a vector whose components are the nodal values on $\Gamma_i$, then

$$\Sigma_i^{-1} q = [0, I] A_i^{-1} [0, I]^T q$$

In particular, the matrix-vector multiplication

$$\begin{pmatrix}
\text{internal nodes} \\
\text{boundary nodes}
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0 \\
q
\end{pmatrix}$$

$$A_i^{-1}$$

corresponds to the solution of a Neumann problem in $\Omega_i$:

$$\begin{cases}
-\Delta w_i = 0 & \text{in } \Omega_i \\
\frac{\partial w_i}{\partial n} = q & \text{in } \Gamma_i
\end{cases}$$
Example

- Condition number of \((P_h^{NN})^{-1}\Sigma\):

<table>
<thead>
<tr>
<th>(\kappa((P_h^{NN})^{-1}\Sigma))</th>
<th>(H = 1/2)</th>
<th>(H = 1/4)</th>
<th>(H = 1/8)</th>
<th>(H = 1/16)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(h = 1/16)</td>
<td>2.55</td>
<td>15.20</td>
<td>47.60</td>
<td>-</td>
</tr>
<tr>
<td>(h = 1/32)</td>
<td>3.45</td>
<td>20.67</td>
<td>76.46</td>
<td>194.65</td>
</tr>
<tr>
<td>(h = 1/64)</td>
<td>4.53</td>
<td>26.25</td>
<td>105.38</td>
<td>316.54</td>
</tr>
<tr>
<td>(h = 1/128)</td>
<td>5.79</td>
<td>31.95</td>
<td>134.02</td>
<td>438.02</td>
</tr>
</tbody>
</table>

- Convergence history of the conjugate gradient using \(P_h^{NN}\) as a preconditioner and \(h = 1/32\):
The Neumann-Neumann preconditioner of the Schur complement system is not scalable.

A substantial improvement can be achieved by adding a coarse grid correction:

\[(P_h^{BNN})^{-1}\Sigma = P_0 + (I - P_0)((P_h^{NN})^{-1}\Sigma)(I - P_0)\]

where \(P_0 = R_0^T\Sigma_0^{-1}R_0\Sigma\) and \(\Sigma_0 = R_0\Sigma R_0^T\), \(R_0\) being the restriction from \(\Gamma\) onto the coarse level skeleton.

This is called balanced Neumann-Neumann preconditioner.

We can prove that

\[\kappa((P_h^{BNN})^{-1}\Sigma) \leq C \left(1 + \log \frac{H}{h}\right)^2\]
Example

<table>
<thead>
<tr>
<th>$\kappa((P_h^{BNN})^{-1} \Sigma)$</th>
<th>$H = 1/2$</th>
<th>$H = 1/4$</th>
<th>$H = 1/8$</th>
<th>$H = 1/16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1/16$</td>
<td>1.67</td>
<td>1.48</td>
<td>1.27</td>
<td>-</td>
</tr>
<tr>
<td>$h = 1/32$</td>
<td>2.17</td>
<td>2.03</td>
<td>1.47</td>
<td>1.29</td>
</tr>
<tr>
<td>$h = 1/64$</td>
<td>2.78</td>
<td>2.76</td>
<td>2.08</td>
<td>1.55</td>
</tr>
<tr>
<td>$h = 1/128$</td>
<td>3.51</td>
<td>3.67</td>
<td>2.81</td>
<td>2.07</td>
</tr>
</tbody>
</table>
CONCLUDING REMARKS

From the numerical results that we have presented, we can conclude with the following remarks:

- even if better conditioned with respect to $A$, $\Sigma$ is ill-conditioned, and therefore a suitable preconditioner must be applied

- the Neumann-Neumann preconditioner can be satisfactory applied using a moderate number of subdomains, while for larger $M$, $\kappa((P_{h}^{NN})^{-1}\Sigma) > \kappa(\Sigma)$

- the balancing Neumann-Neumann preconditioner is almost optimally scalable and therefore recommended for partitions with a large number of subdomains
FETI
(Finite Element Tearing & Interconnecting)
METHODS
We consider the variable coefficient elliptic problem:

\[
\begin{cases}
-\text{div}(\rho \nabla u) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(12)

where \(\rho\) is piecewise constant, \(\rho = \rho_i \in \mathbb{R}^+\) in \(\Omega_i\).

We denote by \(H_i = diam(\Omega_i)\) the size of the \(i\)-th subdomain \(\Omega_i\) for \(i = 1, \ldots, M\).
Domain $\Omega$, its boundary $\partial \Omega$, skeleton $\Gamma$, subdomain $\Omega_i$, and its boundary $\partial \Omega_i$. 
We introduce:

- the space of traces of finite element functions on the boundaries $\partial \Omega_i$, say $W_i = W^h(\partial \Omega_i)$,
- the product space of the previous trace spaces, say $W = \prod_{i=1}^{M} W_i$,
- a subspace of continuous traces across the skeleton $\Gamma$, say $\hat{W} \subset W$.

We denote by $\Omega_{ih}$ the nodes in $\Omega_i$, by $\partial \Omega_{ih}$ the nodes on $\partial \Omega_i$, by $\partial \Omega_h$ the nodes on $\partial \Omega$, and by $\Gamma_h$ the nodes on the skeleton $\Gamma$. 
Finite element degrees of freedom on the boundary ($\partial \Omega_h$), on the skeleton ($\Gamma_h$), in the subdomain ($\Omega_{ih}$), and on its boundary ($\partial \Omega_{ih}$).
Let us introduce the following scaling counting functions:

$$
\delta_i(x) = \begin{cases} 
1 & \text{if } x \in \partial \Omega_{ih} \cap (\partial \Omega_h \setminus \Gamma_h), \\
\left( \sum_{j \in N_x} \rho_j^\gamma(x) \right) / \rho_i^\gamma(x) & \text{if } x \in \partial \Omega_{ih} \cap \Gamma_h, \\
0 & \text{elsewhere,}
\end{cases}
$$

\(\forall x \in \Gamma_h \cup \partial \Omega_h\), where: \(\gamma \in [1/2, +\infty)\) and \(N_x\) is the set of indices of the subregions having \(x\) on their boundaries.

Then we set:

$$
\delta_i^\dagger(x) \quad (= \text{pseudo inverses}) = \begin{cases} 
\delta_i^{-1}(x) & \text{if } \delta_i(x) \neq 0, \\
0 & \text{if } \delta_i(x) = 0.
\end{cases}
$$
THE FINITE ELEMENT APPROXIMATION

Based on the finite element approximation of the model problem (12), let us consider the local Schur complements:

$$
\Sigma_i = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma,\Omega_i} A_{\Omega_i,\Omega_i}^{-1} A_{\Omega_i,\Gamma}, \quad i = 1, \ldots, M,
$$

for which:

$$
\Sigma = R_{\Gamma_1}^T \Sigma_1 R_{\Gamma_1} + \ldots + R_{\Gamma_M}^T \Sigma_M R_{\Gamma_M},
$$

where $R_{\Gamma_i}$ is a restriction operator, that is a rectangular matrix of zeros and ones that map values on $\Gamma$ onto those on $\Gamma_i$, $i = 1, \ldots, M$. The matrices $\Sigma_i$ are positive semi-definite.

We indicate the interface nodal values on $\partial \Omega_i$ as $u_i$, and we set $u = (u_1, \ldots, u_M)$, the local load vectors on $\partial \Omega_i$ as $\chi_i$ and we set $\chi_\Delta = (\chi_1, \ldots, \chi_M)$. Finally, we set:

$$
\Sigma_\Delta = \text{diag}(\Sigma_1, \ldots, \Sigma_M),
$$

a block diagonal matrix.
THE REDUCED FEM PROBLEM

The original FEM problem, when reduced to the interface $\Gamma$, reads:

\[
\begin{align*}
\text{Find } u \in W \text{ s.t. } J(u) &= \frac{1}{2} \langle \Sigma \Delta u, u \rangle - \langle \chi_{\Delta}, u \rangle \rightarrow \min, \\
B_\Gamma u &= 0.
\end{align*}
\] (13)

The reduced problem (13) admits a unique solution iff $\text{Ker}\{\Sigma_{\Delta}\} \cap \text{Ker}\{B_\Gamma\} = 0$, that is if $\Sigma_{\Delta}$ is invertible on $\text{Ker}(B_\Gamma)$.

The matrix $B_\Gamma$ is not unique, so that we should impose continuity when $u$ belongs to more than one subdomain; $B_\Gamma$ is made of $\{0, -1, 1\}$ since it enforces interfaces continuity constraints at interfaces nodes.

**Remark**
We are using the same notation $W$ to denote the finite element space trace and that of their nodal values at points of $\Gamma_h$. 
THE CHOICE OF THE MATRIX $B_\Gamma$

In 2D, there is a little choice on how to write the constraint of continuity at a point sitting on an edge, there are many options for a vertex point.

For the edge node we only need to choose the sign, whereas for a vertex node, e.g. one common to 4 subdomains, a minimum set of three constraints can be chosen in many different ways to assure continuity at the node in question.

Continuity constraints enforced by 3 (non-redundant) conditions on the left, by 6 (redundant) conditions on the right.
THE REDUCED FEM PROBLEM WITH LAGRANGE MULTIPLIERS

We reformulate the reduced problem (13) by introducing suitable Lagrange multipliers:

\[
\begin{align*}
\text{Find } (u, \lambda) \in W \times U & \text{ s.t. } \Sigma \Delta u + B^T \lambda = \chi_{\Delta}, \\
B \Gamma u & = 0.
\end{align*}
\]

(14)

Because of the inf-sup (LBB) condition, the component \( \lambda \) of the solution to (14) is unique up to an additive vector of \( \text{Ker}(B^T \Gamma) \), so we choose \( U = \text{range}(B \Gamma) \).
Let the matrix $R = \text{diag} \left( R^{(1)}, \ldots, R^{(M)} \right)$ be made of null-space elements of $\Sigma_{\Delta}$. (E.g. $R^{(i)}$ corresponds to the rigid body motions of $\Omega_i$, in case of linear elasticity operator.)

$R$ is a full column rank matrix.

The solution of the first equation of the reduced problem (14) exists iff $\chi_{\Delta} - B_{\Gamma}^T \lambda \in \text{range}(\Sigma_{\Delta})$, a limitation that will be resolved by introducing a suitable projection operator $P$. Then,

\[ u = \Sigma_{\Delta}^\dagger (\chi_{\Delta} - B_{\Gamma}^T \lambda) + R\alpha \quad \text{if} \quad \chi_{\Delta} - B_{\Gamma}^T \lambda \perp \text{Ker}(\Sigma_{\Delta}), \]

where $\alpha$ is an arbitrary vector and $\Sigma_{\Delta}^\dagger$ a pseudoinverse of $\Sigma_{\Delta}$. 
Substituting $u$ into the second equation of the reduced problem (14) yields:

$$B_{\Gamma} \Sigma_{\Delta}^{\dagger} B_{\Gamma}^{T} \lambda = B_{\Gamma} \Sigma_{\Delta}^{\dagger} \chi_{\Delta} + B_{\Gamma} R_{\alpha}. \quad (15)$$

Let us set $F = B_{\Gamma} \Sigma_{\Delta}^{\dagger} B_{\Gamma}^{T}$ and $d = B_{\Gamma} \Sigma_{\Delta}^{\dagger} \chi_{\Delta}$.

Then choose $P^{T}$ to be a suitable projection matrix, e.g. $P^{T} = I - G(G^{T}G)^{-1}G^{T}$, with $G = B_{\Gamma} R$.

We obtain:

$$\begin{cases} P^{T} F \lambda & = P^{T} d, \\ G^{T} \lambda & = e \ (= R^{T} \chi_{\Delta}). \end{cases}$$
The original one-level FETI method is a CG method in the space $V$ applied to:

$$P^T F \lambda = P^T d, \quad \lambda \in \lambda_0 + V,$$

with an initial $\lambda_0$ s.t. $G^T \lambda_0 = e$. Here

$$V = \{ \lambda \in U : \langle \lambda, B_{\Gamma} z \rangle = 0, z \in Ker(\Sigma_{\Delta}) \}$$

is the so-called space of admissible increments, $Ker(G^T) = range(P)$ and:

$$V' = \{ \mu \in U : \langle \mu, B_{\Gamma} z \rangle_Q = 0, z \in Ker(\Sigma_{\Delta}) \} = range(P^T).$$
The original, most basic FETI preconditioner is:

\[ P_h^{-1} = B_\Gamma \Sigma_\Delta B_\Gamma^T = \sum_{i=1}^{M} B^{(i)} \Sigma_i B^{(i)^T} \].

It is called a Dirichlet preconditioner since its application to a given vector involves the solution of \( M \) independent Dirichlet problems, one in every subdomain.

The coarse space in FETI consists of the nullspace on each substructure.
In case $B_Γ$ has full row rank (i.e. the constraints are linearly independent and there are no redundant Lagrange multipliers), a better preconditioner can be defined as follows:

$$
\hat{P}_h^{-1} = (B_Γ D^{-1} B_Γ^T)^{-1} B_Γ D^{-1} \Sigma_\Delta D^{-1} B_Γ^T (B_Γ D^{-1} B_Γ^T)^{-1},
$$

where $D$ is a block diagonal matrix with positive entries $D = \text{diag} \left( D^{(1)}, \ldots, D^{(M)} \right)$ with $D^{(i)}$ a diagonal matrix whose elements are $\delta_i^\dagger(x)$ corresponding to the point $x \in \partial \Omega_{ih}$ (recall the scaling counting functions).
Remark
Since $B_{\Gamma} D^{-1} B_{\Gamma}^T$ is block-diagonal, its inverse can be easily computed by inverting small blocks whose size is $n_x$, the number of Lagrange multipliers used to enforce continuity at point $x$.

The matrix $D$, that operates on elements of the product space $W$, can be regarded as a scaling from the right of $B_{\Gamma}$ by $D^{-1/2}$. With this choice:

$$K_2(P \hat{P}_h^{-1} P^T F) \leq C(1 + \log(H/h))^2,$$

where $K_2(\cdot)$ is the spectral condition number and $C$ is a constant independent of $h$, $H$, $\gamma$ and the values of the $\rho_i$. 
FETI-DP
(Dual Primal FETI)
METHODS
OVERVIEW OF THE FETI-DP METHODS

- The FETI-DP method is a domain decomposition method that enforces equality of the solution at subdomains interfaces by Lagrange multipliers except at subdomains corners, which remain primal variables.

- The first mathematical analysis of the method was provided by Mandel and Tezaur (2001).

- The method was further improved by enforcing the equality of averages across the edges or faces on subdomain interfaces. This is important for parallel scalability.
FINITE ELEMENT SPACES

We consider a 2D case for simplicity. We introduce a space \( \tilde{W} \) s.t. \( \hat{W} \subset \tilde{W} \subset W \) for which we have continuity of the primal variables at subdomain vertices. We write \( \tilde{W} \) as:

\[
\tilde{W} = \hat{W}_\Pi \oplus \tilde{W}_\Delta,
\]

where:

- \( \hat{W}_\Pi \subset \hat{W} \) is the space of continuous interface functions that vanish at all nodal points of \( \Gamma_h \) except at the subdomain vertices. \( \hat{W}_\Pi \) is given in terms of the vertex variables and the averages of the values over the individual edges of the set of interface nodes \( \Gamma_h \).

- \( \tilde{W}_\Delta \) is the direct sum of local subspaces \( \tilde{W}_{\Delta,i} \): \( \tilde{W}_\Delta = \prod_{i=1}^{M} \tilde{W}_{\Delta,i} \), where \( \tilde{W}_{\Delta,i} \subset W_i \) consists of local functions on \( \partial \Omega_i \) that vanish at the vertices of \( \Omega_i \) and have zero average on each individual edge.
According to this space splitting:

- the continuous degrees of freedom associated with the subdomain vertices and with the subspace $\tilde{W}_\Pi$ are called \textit{primal} ($\Pi$);

- those (that are potentially discontinuous across $\Gamma$) that are associated with the subspaces $\tilde{W}_{\Delta,i}$ and with the interior of the subdomain edges are called \textit{dual} ($\Delta$).

The subspace $\tilde{W}_\Pi$, together with the interior subspace, defines the subsystem which is fully assembled, factored, and stored in each iteration step.
All unknowns of the subspace $\widehat{W}_Π$ as well as the interior variables are eliminated to obtain a new Schur complement $\tilde{\Sigma}_\Delta$.

We consider the following procedure.

Let $\tilde{A}$ denote the stiffness matrix obtained by restricting $\text{diag} (A_1, \ldots, A_M)$ from $\prod_{i=1}^M W^h(\Omega_i)$ to $\widehat{W}^h(\Omega)$ (these spaces now refer to subdomains, not their boundaries).

Then $\tilde{A}$ is no longer block diagonal because of the coupling that now exists between subdomains that share a common vertex.
According to the previous *space decomposition*, $\tilde{A}$ can be split as:

$$
\tilde{A} = \begin{bmatrix}
A_{II} & A_{I\Pi} & A_{I\Delta} \\
A_{I\Pi}^T & A_{\Pi\Pi} & A_{\Pi\Delta} \\
A_{I\Delta}^T & A_{\Pi\Delta}^T & A_{\Delta\Delta}
\end{bmatrix}.
$$

The subscript $I$ refers to the *internal degrees of freedom* of the subdomains, $\Pi$ to those associated to the subdomains vertices, and $\Delta$ to those of the interior of the subdomains edges.

The matrices $A_{II}$ and $A_{\Delta\Delta}$ are block diagonal (one block per subdomain). Any non-zero entry of $A_{I\Delta}$ represents a coupling between degrees of freedom associated with the same subdomain.
Degrees of freedom of the space $\mathcal{W}$ for one-level FETI (left) and those of the space $\tilde{\mathcal{W}}$ for one-level FETI-DP (right) in the case of primal vertices only.
Upon eliminating the variables of the $I$ and $\Pi$ sets, a Schur complement associated with the variables of the $\Delta$ sets (interior and edges) is obtained as follows:

\[
\tilde{\Sigma} = \begin{bmatrix} \mathbf{A}_{\Delta \Delta} - [\mathbf{A}_{I \Delta}^{T} \mathbf{A}_{\Pi \Delta}] \begin{bmatrix} \mathbf{A}_{I I} & \mathbf{A}_{I \Pi} \\ \mathbf{A}_{I I}^{T} & \mathbf{A}_{\Pi \Pi} \end{bmatrix}^{-1} & \mathbf{A}_{I \Delta} \\ \mathbf{A}_{I \Pi} & \mathbf{A}_{\Pi \Pi} \end{bmatrix} \end{bmatrix}.
\]

Correspondingly we obtain a reduced right hand side $\tilde{\chi}_{\Delta}$. 
THE REDUCED FEM PROBLEM

By indicating with \( \mathbf{u}_\Delta \in \tilde{\mathcal{W}}_\Delta \) the vector of degrees of freedom associated with the edges, similarly to what done in (13) for FETI, the finite element problem can be reformulated as a reduced minimization problem with constraints given by the requirement of continuity across all of \( \Gamma \):

\[
\begin{align*}
\text{Find } \mathbf{u}_\Delta \in \tilde{\mathcal{W}}_\Delta : \quad & J(\mathbf{u}_\Delta) = \frac{1}{2} \langle \tilde{\Sigma} \mathbf{u}_\Delta, \mathbf{u}_\Delta \rangle - \langle \tilde{\chi}_\Delta, \mathbf{u}_\Delta \rangle \to \min, \\
& B_\Delta \mathbf{u}_\Delta = 0.
\end{align*}
\]

The matrix \( B_\Delta \) is made of \( \{0, -1, 1\} \) as it was for \( B_\Gamma \).

**Remark**
The constraints associated with the vertex nodes are dropped since they are assigned to the primal set.

**Remark**
Since all the constraints refer to edge points, no distinction needs to be made between redundant and non-redundant constraints and Lagrange multipliers.
A saddle point formulation of (16), similar to (14), can be obtained by introducing a set of Lagrange multipliers \( \lambda \in V = \text{range}(B_{\Delta}) \).

Indeed, since \( \tilde{A} \) is s.p.d., so is \( \tilde{\Sigma} \): by eliminating the subvectors \( u_{\Delta} \) we obtain the reduced system:

\[
F_{\Delta} \lambda = d_{\Delta},
\]

where \( F_{\Delta} = B_{\Delta} \tilde{\Sigma}^{-1} B_{\Delta}^T \) and \( d_{\Delta} = B_{\Delta} \tilde{\Sigma}^{-1} \tilde{\chi}_{\Delta} \).

**Remark**
Once \( \lambda \) is found, \( u_{\Delta} = \tilde{\Sigma}^{-1}(\tilde{\chi}_{\Delta} - B_{\Delta}^T \lambda) \in \tilde{W}_{\Delta} \), while the interior variables \( u_I \) and the vertex variables \( u_{\Pi} \) are obtained by back-solving the system associated with \( \tilde{A} \).
A FETI-DP PRECONDITIONER

A preconditioner for $F$ is introduced as done for FETI (in case of non-redundant Lagrange multipliers):

$$P^{-1}_\Delta = (B_\Delta D_\Delta^{-1} B_\Delta^T)^{-1} B_\Delta D_\Delta^{-1} S_{\Delta \Delta} D_\Delta^{-1} B_\Delta^T (B_\Delta D_\Delta^{-1} B_\Delta^T)^{-1}.$$  

- $D_\Delta$ is a diagonal scaling matrix with blocks $D_\Delta^{(i)}$: each of their diagonal elements corresponds to a Lagrange multiplier that enforces continuity between the nodal values of some $w_i \in \mathcal{W}_i$ and $w_j \in \mathcal{W}_j$ at some point $x \in \Gamma_h$ and it is given by $\delta_j^\dagger(x)$;

- $S_{\Delta \Delta} = \text{diag} \left( \Sigma_{1,\Delta \Delta}, \ldots, \Sigma_{M,\Delta \Delta} \right)$ with $\Sigma_{i,\Delta \Delta}$ being the restriction of the local Schur complement $\Sigma_i$ to $\tilde{\mathcal{W}}_{\Delta,i} \subset \mathcal{W}_i$. 
Remark
When using the conjugate gradient method for the preconditioned system:

\[ P_\Delta^{-1} F_\Delta \lambda = P_\Delta^{-1} d_\Delta, \]

in contrast with one level FETI methods we can use an arbitrary initial guess \( \lambda^0 \).

We have a condition number that scales polylogarithmically, that is:

\[ K_2(P_\Delta^{-1} F_\Delta) \leq C(1 + \log(H/h))^2, \]

where \( C \) is independent of \( h, H, \gamma \) and the values of the \( \rho_j \).
COMPARISON OF FETI AND FETI-DP METHODS

- **FETI-DP** algorithms do not require the characterization of the kernels of local Neumann problems (as required by one-level FETI methods), because the enforcement of the additional constraints in each iteration always makes the local problems nonsingular and at the same time provides an underlying coarse global problem.

- **FETI-DP** methods do not require the introduction of a scaling matrix $Q$, which enters in the construction of a coarse solver for one-level FETI algorithms.

- **One-level FETI** methods are projected conjugate gradient algorithms that cannot start from an arbitrary initial guess. In contrast, FETI-DP methods are standard preconditioned conjugate algorithms and can therefore employ an arbitrary initial guess $\lambda^0$. 
BDDC
(Balancing Domain Decomposition with Constraints)
METHODS
A QUICK VIEW OF BDDC METHODS

- The BDDC method was introduced by *Dohrmann* (2003) as a simpler primal alternative to the FETI-DP domain decomposition method.

- The name BDDC was coined by *Mandel* and *Dohrmann* because it can be understood as further development of the balancing domain decomposition method.

- BDDC is used as a preconditioner for the conjugate gradient method.
A specific version of BDDC is characterized by the choice of **coarse degrees of freedom**, which can be values at the corners of the subdomains, or averages over the edges of the interface between the subdomains.

One application of the BDDC preconditioner **combines** the solution of **local problems** on each **subdomain** with the solution of a **global coarse problem** with the coarse degrees of freedom as the unknowns. The local problems on different subdomains are completely independent of each other, so the method is suitable for **parallel computing**.
A BDDC preconditioner reads:

\[
P^{-1}_{BDDC} = \tilde{R}_D^T \tilde{\Sigma}^{-1} \tilde{R}_D,
\]

where \( \tilde{R}_\Gamma : \tilde{\mathcal{W}} \to \tilde{\mathcal{W}} \) is a restriction matrix, \( \tilde{R}_{DG} \) is a scaled variant of \( \tilde{R}_\Gamma \) with scale factor \( \delta_i^\dagger \) (featuring the same sparsity pattern of \( \tilde{R}_\Gamma \)).

This scaling is chosen in such a way that \( \tilde{R}_\Gamma \tilde{R}_D^T \) is a projection (then it coincides with its square).
ALGEBRAIC FORM OF SCHWARZ ITERATIVE METHODS
ALGEBRAIC FORM OF SCHWARZ METHODS FOR FE DISCRETIZATION

- Let $A\mathbf{u} = \mathbf{f}$ be the system associated to the finite element approximation of the Poisson problem.
- We still assume that $\Omega$ is decomposed in two overlapping subdomains $\Omega_1$ and $\Omega_2$.
- We denote by $N_h$ the total number of interior nodes of $\Omega$, and by $n_1$ and $n_2$ the interior nodes of $\Omega_1$ and $\Omega_2$: $N_h \leq n_1 + n_2$
- The stiffness matrix $A$ contains two submatrices, say $A_1$ and $A_2$, which correspond to the local stiffness matrices associated to the Dirichlet problems in $\Omega_1$ and $\Omega_2$:

\[
A = \begin{bmatrix}
A_1 & \text{(interior nodes)} \\
\text{(interior nodes)} & A_2
\end{bmatrix}
\begin{bmatrix}
n_1 \\
n_2
\end{bmatrix}
\begin{bmatrix}
N_h
\end{bmatrix}
\]
MULTIPLICATIVE SCHWARZ METHOD

Consider the multiplicative method: given \( u_2^{(0)} \) on \( \Gamma_1 \), for \( k \geq 1 \):

\[
\begin{cases}
Lu_1^{(k)} = f & \text{in } \Omega_1 \\
u_1^{(k)} = u_2^{(k-1)} & \text{on } \Gamma_1 \\
u_1^{(k)} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma_1
\end{cases}
\]

\[
\begin{pmatrix}
A_{\Omega_1} & A_{\Gamma_1}
\end{pmatrix}
\begin{pmatrix}
u_{\Omega_1}^{(k)} \\
u_{\Gamma_1}^{(k)}
\end{pmatrix} = f_1
\]

\[
u_{\Gamma_1}^{(k)} = u_{\Omega_2|\Gamma_1}^{(k-1)}
\]

\[
\begin{cases}
Lu_2^{(k)} = f & \text{in } \Omega_2 \\
u_2^{(k)} = u_1^{(k)} & \text{on } \Gamma_2 \\
u_2^{(k)} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma_2
\end{cases}
\]

\[
\begin{pmatrix}
A_{\Omega_2} & A_{\Gamma_2}
\end{pmatrix}
\begin{pmatrix}
u_{\Omega_2}^{(k)} \\
u_{\Gamma_2}^{(k)}
\end{pmatrix} = f_2
\]

\[
u_{\Gamma_2}^{(k)} = u_{\Omega_1|\Gamma_2}^{(k)}
\]
We find:

\[
A_{\Omega_1} u^{(k)}_{\Omega_1} = f_1 - A_{\Gamma_1} u^{(k-1)}_{\Gamma_1}
\]
\[
A_{\Omega_2} u^{(k)}_{\Omega_2} = f_2 - A_{\Gamma_2} u^{(k)}_{\Gamma_2}
\]

or, equivalently, by adding and subtracting \( u^{(k-1)}_{\Omega_i} \):

\[
u^{(k)}_{\Omega_1} = u^{(k-1)}_{\Omega_1} + A^{-1}_{\Omega_1}(f_1 - A_{\Omega_1} u^{(k-1)}_{\Omega_1} - A_{\Gamma_1} u^{(k-1)}_{\Gamma_1})
\]
\[
u^{(k)}_{\Omega_2} = u^{(k-1)}_{\Omega_2} + A^{-1}_{\Omega_2}(f_2 - A_{\Omega_2} u^{(k-1)}_{\Omega_2} - A_{\Gamma_2} u^{(k)}_{\Gamma_2})
\]

**Remark**

This method can be seen as a block Gauss-Seidel method to solve the system

\[
\begin{pmatrix}
A_{\Omega_1} & A_{\Gamma_1} \\
A_{\Gamma_2} & A_{\Omega_2}
\end{pmatrix}
\begin{pmatrix}
u_{\Omega_1} \\
u_{\Omega_2}
\end{pmatrix}
= \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}
\]
If there is no direct coupling between the matrices assembled on opposite sides of $\Gamma_1$ and $\Gamma_2$:

$$A_{\Omega_2 \setminus \Omega_1} = 0 \quad \text{and} \quad A_{\Omega_1 \setminus \Omega_2} = 0$$

then, formally, we can write

$$A_{\Gamma_1} = A_{(\Omega \setminus \Omega_1) \setminus (\Omega_2 \setminus \Omega_1)} = A_{(\Omega \setminus \Omega_1)}$$

so that

$$A_{\Gamma_1} u_{\Gamma_1}^{(k-1)} = A_{(\Omega \setminus \Omega_1)} u_{(\Omega \setminus \Omega_1)}^{(k-1)}$$

Proceeding in a similar way also for the second equation, we finally obtain:

$$u_{\Omega_1}^{(k)} = u_{\Omega_1}^{(k-1)} + A_{\Omega_1}^{-1} (f_1 - A_{\Omega_1} u_{\Omega_1}^{(k-1)} - A_{(\Omega \setminus \Omega_1)} u_{(\Omega \setminus \Omega_1)}^{(k-1)})$$

$$u_{\Omega_2}^{(k)} = u_{\Omega_2}^{(k-1)} + A_{\Omega_2}^{-1} (f_2 - A_{\Omega_2} u_{\Omega_2}^{(k-1)} - A_{(\Omega \setminus \Omega_2)} u_{(\Omega \setminus \Omega_2)}^{(k-1)})$$
In compact form we can write

\[
\begin{align*}
\mathbf{u}^{(k+\frac{1}{2})} &= \mathbf{u}^{(k)} + \begin{pmatrix} A_{\Omega_1}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \left( \mathbf{f} - A\mathbf{u}^{(k)} \right) \\
\mathbf{u}^{(k+1)} &= \mathbf{u}^{(k+\frac{1}{2})} + \begin{pmatrix} 0 & 0 \\ 0 & A_{\Omega_2}^{-1} \end{pmatrix} \left( \mathbf{f} - A\mathbf{u}^{(k+\frac{1}{2})} \right)
\end{align*}
\]

Let \( R_i \) be the (rectangular) restriction matrix that returns the vector of coefficients in the interior of \( \Omega_i \):

\[
\mathbf{u}_{\Omega_i} = R_i \mathbf{u} = (I \ 0) \begin{pmatrix} \mathbf{u}_{\Omega_i} \\ \mathbf{u}_{\Omega \setminus \Omega_1} \end{pmatrix}
\]

Then, we obtain:

\[
\begin{align*}
\mathbf{u}^{(k+\frac{1}{2})} &= \mathbf{u}^{(k)} + R_1^T (R_1 A R_1^T)^{-1} R_1 \left( \mathbf{f} - A\mathbf{u}^{(k)} \right) \\
\mathbf{u}^{(k+1)} &= \mathbf{u}^{(k+\frac{1}{2})} + R_2^T (R_2 A R_2^T)^{-1} R_2 \left( \mathbf{f} - A\mathbf{u}^{(k+\frac{1}{2})} \right)
\end{align*}
\]
Define, for $i = 1, 2$

$$Q_i = R_i^T (R_iAR_i^T)^{-1} R_i$$

We get

$$u^{(k+\frac{1}{2})} = u^{(k)} + Q_1 \left( f - Au^{(k)} \right)$$

$$u^{(k+1)} = u^{(k+\frac{1}{2})} + Q_2 \left( f - Au^{(k+\frac{1}{2})} \right)$$

and

$$u^{(k+1)} = u^{(k)} + Q_1 \left( f - Au^{(k)} \right) + Q_2 \left[ f - A \left( u^{(k)} + Q_1 \left( f - Au^{(k)} \right) \right) \right]$$

or, equivalently,

$$u^{(k+1)} = u^{(k)} + \left( Q_1 + Q_2 - Q_2AQ_1 \right) \left( f - Au^{(k)} \right)$$
ADDITIVE SCHWARZ METHOD

- Proceeding as before, we can prove that the additive Schwarz method can be seen as a block Jacobi method to solve the system

\[
\begin{pmatrix}
A_{\Omega_1} & A_{\Gamma_1} \\
A_{\Gamma_2} & A_{\Omega_2}
\end{pmatrix}
\begin{pmatrix}
u_{\Omega_1} \\
u_{\Omega_2}
\end{pmatrix} = \begin{pmatrix}f_1 \\ f_2\end{pmatrix}
\]

- One iteration of the additive Schwarz method can be written as

\[
u^{(k+1)} = \nu^{(k)} + (Q_1 + Q_2)(f - A\nu^{(k)})
\]

- In the case of a \( M \geq 2 \) overlapping subdomains \( \{\Omega_i\} \) we have:

\[
u^{(k+1)} = \nu^{(k)} + \left(\sum_{i=1}^{M} Q_i\right) (f - A\nu^{(k)})
\]
THE SCHWARZ METHOD AS A PRECONDITIONER

- **Multiplicative Schwarz method**: can be interpreted as a Richardson iterative procedure for $Au = f$ with the multiplicative Schwarz preconditioner

  $$P^{-1}_{ms} = Q_1 + Q_2 - Q_2AQ_1$$

  $P_{ms}$ is not symmetric and can be used within GMRES or BiCGStab iterations.

- **Additive Schwarz method**: corresponds to Richardson iterations for $Au = f$ with the additive Schwarz preconditioner

  $$P_{as} = \left( \sum_{i=1}^{M} Q_i \right)^{-1}$$
The convergence rate for the additive Schwarz method is slower than for the multiplicative Schwarz method. In the case of two subdomains, there is approximately a factor $\frac{1}{2}$. (This is similar to the classical convergence results for the Jacobi and Gauss-Seidel methods)

$P_{as}$ is not optimal as the condition number blows up if the size of the subdomains reduces:

$$\kappa(P_{as}^{-1}A) \leq C \frac{1}{\delta H}$$

where $C$ is a constant independent of $h$, $H$, and $\delta$, $\delta$ being a linear measure of the overlapping region(s) ($h \lesssim \delta \lesssim H$).
ALGORITHMIC ASPECTS & NUMERICAL RESULTS

- **Create overlapping subdomains**
  - a) Triangulate the computational domain $\Omega \Rightarrow \mathcal{T}_h$.
  - b) Partition the mesh into $M$ nonoverlapping subdomains $\{\hat{\Omega}_i\}_{i=1}^M$.
  - c) Extend every subdomain $\hat{\Omega}_i$ by adding all the strips of finite elements of $\mathcal{T}_h$ within distance $\delta$ from $\hat{\Omega}_i$.

- **Start-up phase for the application of $P_{as}$**
  - a) On each subdomain $\Omega_i$ “build” $R_i$ and $R_i^T$.
  - b) Build the matrix $A$ corresponding to the FE discretization on $\mathcal{T}_h$.
  - c) On each $\Omega_i$ form the local submatrices $A_{\Omega_i} = R_i A R_i^T$.
  - d) On each $\Omega_i$ prepare the code to solve a linear system with $A_{\Omega_i}$: compute a factorization (LU, ILU, Cholesky...) of $A_{\Omega_i}$. 

**One-level additive Schwarz preconditioner** $P_{as}$

Given a vector $r$, compute $z = P_{as}^{-1}r$ as follows

a) set $z = 0$;
   For $i = 1, \ldots, M$, do in parallel
b) restrict the residual over $\Omega_i \setminus \partial \Omega_i$: $r_i = R_i r$;
c) solve $A_{\Omega_i} z_i = r_i$;
d) add to the global residual: $z \leftarrow z + R_i^T z_i$.
   EndFor

**Example**

<table>
<thead>
<tr>
<th>$\kappa(P_{as}^{-1}A)$</th>
<th>$H = 1/2$</th>
<th>$H = 1/4$</th>
<th>$H = 1/8$</th>
<th>$H = 1/16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1/16$</td>
<td>15.95</td>
<td>27.09</td>
<td>52.08</td>
<td>-</td>
</tr>
<tr>
<td>$h = 1/32$</td>
<td>31.69</td>
<td>54.52</td>
<td>104.85</td>
<td>207.67</td>
</tr>
<tr>
<td>$h = 1/64$</td>
<td>63.98</td>
<td>109.22</td>
<td>210.07</td>
<td>416.09</td>
</tr>
<tr>
<td>$h = 1/128$</td>
<td>127.99</td>
<td>218.48</td>
<td>420.04</td>
<td>832.57</td>
</tr>
</tbody>
</table>
The main limitation of Schwarz methods is to propagate information only among neighboring subdomains, since only local solves are involved by the application of $(P_{as})^{-1}$.

→ We have to introduce a “coarse” global problem over the whole domain to guarantee a mechanism of global communication among all subdomains.
TWO-LEVEL SCHWARZ PRECONDITIONERS

As for the Neumann-Neumann method, we can introduce a coarse-grid mechanism that allows for a sudden information diffusion on the whole domain $\Omega$:

- consider the subdomains as macro-elements of a new coarse grid $\mathcal{T}_H$
- build a corresponding stiffness matrix $A_H$
- The matrix $Q_0 = R_H^T A_H^{-1} R_H$ represents the coarse level correction for the two-level preconditioner, with $R_H$ the restriction operator from the fine to the coarse grid.

The two-level preconditioner $P_{\text{cas}}$ is defined as:

$$ P_{\text{cas}}^{-1} = \sum_{i=0}^{M} Q_i $$
We can prove that there exists a constant $C > 0$, independent of both $h$ and $H$ such that

$$\kappa(P_{\text{cas}}^{-1}A) \leq C \left(1 + \frac{H}{\delta}\right)$$

If $\delta$ is a fraction of $H$ (generous overlap), the preconditioner $P_{\text{cas}}$ is scalable and optimal.

If $\delta \simeq h$, the preconditioner $P_{\text{cas}}$ is neither scalable or optimal.

- Iterations on the original finite element system using $P_{\text{cas}}$ converge with a rate independent of $h$ and $H$ (and therefore of the number of subdomains)

- Thanks to the additive structure of the preconditioner, the preconditioning step is fully parallel as it involves the solution of independent systems, one per each local matrix $A_{\Omega_i}$. 
ALGORITHMIC ASPECTS & NUMERICAL RESULTS

- **Start-up phase to apply** $P_{cas}$

  a) Do the start-up phase for $P_{as}$.
  b) Define a coarse space triangulation $\mathcal{T}_0$ with elements of order $H$ and $n_0$ dofs:
  
  c) Build the restriction operator $R_H \in \mathbb{R}^{n_0 \times N_h}$, $R_H(i,j) = \varphi_i(x_j)$. $\varphi_i$ is the basis function associated to the coarse grid node $i$ and $x_j$ are the coordinates of the fine grid node $j$.
  d) Construct the coarse matrix $A_H$, either as

  $$A_H(i,j) = \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j \quad \text{or} \quad A_H = R_H A R_H^T$$
Two-levels additive Schwarz preconditioner $P_{\text{cas}}$

Given a vector $r$, compute $z = P_{\text{cas}}^{-1}r$ as follows

a) set $z = 0$;
   For $i = 1, \ldots, M$, do in parallel

b) restrict the residual over $\Omega_i \setminus \partial \Omega_i$: $r_i = R_i r$;

c) solve $A_{\Omega_i} z_i = r_i$;

d) add to the global residual: $z \leftarrow z + R_i^T z_i$.
EndFor

e) Compute the coarse-grid contribution $z_H = A_H^{-1}(R_H r)$;

f) Add to the global residual $z \leftarrow z + R_H^T z_H$.

Example

<table>
<thead>
<tr>
<th>$\kappa (P_{\text{cas}}^{-1} A)$</th>
<th>$H = 1/4$</th>
<th>$H = 1/8$</th>
<th>$H = 1/16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1/32$</td>
<td>7.03</td>
<td>4.94</td>
<td>-</td>
</tr>
<tr>
<td>$h = 1/64$</td>
<td>12.73</td>
<td>7.59</td>
<td>4.98</td>
</tr>
<tr>
<td>$h = 1/128$</td>
<td>23.62</td>
<td>13.17</td>
<td>7.66</td>
</tr>
<tr>
<td>$h = 1/256$</td>
<td>45.33</td>
<td>24.34</td>
<td>13.28</td>
</tr>
</tbody>
</table>

If $H/\delta = \text{constant}$, this two-level preconditioner is either optimal and scalable.
Practical indications:

- For decompositions with a small number of subdomains, the single level Schwarz preconditioner $P_{as}$ is very efficient.
- When the number $M$ of subdomains gets large, using two-level preconditioners becomes crucial.

When generating a coarse grid is difficult, other algebraic techniques, like aggregation, can be adopted to define $A_H$, and

\[ \kappa(P_{agg}^{-1} A) \leq C \left( 1 + \frac{H}{\delta} \right) \]

<table>
<thead>
<tr>
<th>$P_{agg}^{-1}$</th>
<th>$H = 1/4$</th>
<th>$H = 1/8$</th>
<th>$H = 1/16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1/16$</td>
<td>13.37</td>
<td>8.87</td>
<td>-</td>
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<tr>
<td>$h = 1/32$</td>
<td>26.93</td>
<td>17.71</td>
<td>9.82</td>
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<tr>
<td>$h = 1/64$</td>
<td>54.33</td>
<td>35.21</td>
<td>19.70</td>
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<tr>
<td>$h = 1/128$</td>
<td>109.39</td>
<td>70.22</td>
<td>39.07</td>
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</tbody>
</table>
References


Official page of Domain Decomposition Methods

- www.ddm.org