Supplementary Notes for:

Nonlinear elasticity with applications to the arterial wall

Short Course in the
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Notation

Before proceeding further, we discuss some of the notation used in this chapter. We make use of Cartesian coordinates and the standard Einstein summation convention. For clarity, relations are often given in both coordinate free notation as well as component form.

In general, lower case letters (Greek and Latin) are used for scalar quantities, boldface lowercase letters are used for first order tensors (vectors) and boldface upper case letters are used for second order tensors. The components of tensors relative to fixed orthonormal basis (rectilinear) $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ are denoted using Latin subscripts. Standard summation convention is used whereby repeated Latin indices imply summation from one to three. The inner product between two arbitrary vectors $\underline{u}$ and $\underline{v}$ is denoted using the "$\cdot$" notation and defined in this chapter as,

$$\underline{u} \cdot \underline{v} \;=\; u_i\, v_i. \tag{1}$$

The linear transformation formed by the operation of a second order tensor $\underline{A}$ on a vector $\underline{u}$ to generate a vector $\underline{v}$ is written as $\underline{v} = \underline{Au}$ [1]. The product of two second order tensors $\underline{A}\,\underline{B}$ generates another second order tensor $\underline{C}$,

$$\underline{C} \;=\; \underline{AB} \qquad \text{or} \qquad C_{ij} \;=\; A_{ik}\, B_{kj}. \tag{2}$$

The inner product of two second order tensors $\underline{A}$ and $\underline{B}$ is denoted by $\underline{A} : \underline{B}$ and defined as,

$$\underline{A} : \underline{B} \;=\; \text{trace}(\underline{A}^T\, \underline{B}) \qquad \text{or} \qquad \underline{A} : \underline{B} \;=\; A_{ij}\, B_{ij}. \tag{3}$$

[1] Some texts write $\underline{v} = \underline{A}\,\underline{u}$.

Chapter 1

Kinematics

1.1 Continuum Idealization

In this class, we consider the bulk response of materials, rather than focusing on the behavior at the microscopic level. In doing so, we will make use of a continuum mechanics model in which we assume that the physical body under analysis completely fills a region in physical space. Namely, we idealize a physical body, $\mathfrak{B}$, as being composed of a continuous set of material points (or continuum particles).

1.2 Description of motion of material points in a body

In order to identify these material points within the body and describe the motion/deformation of the body, we embed $\mathfrak{B}$ in a three-dimensional Euclidean space, Figure 1.2. As the body deforms (moves) in time, it will occupy different regions in space. We call these regions configurations and denote the configuration at an arbitrary current time t by $\kappa(t)$. This configuration will be referred to as the *current configuration*. We choose a particular configuration of $\mathfrak{B}$ to serve as a *reference configuration*, enabling us to uniquely identify an arbitrary material point P in $\mathfrak{B}$ by its position $\underline{X}$ in this configuration. We denote this special configuration as κ_0 [1]. For example, κ_0 could be the configuration of the body at time zero, though it is not necessary that the configuration of the body have ever coincided with reference configuration κ_0.

During the deformation (or motion) of $\mathfrak{B}$, an arbitrary material particle P located at position $\underline{X}$ in reference configuration κ_0 will move to position $\underline{x}$ in configuration $\kappa(t)$. The vector $\underline{X}$ is sometimes referred to as the *referential position* and $\underline{x}$ as the *current position* of the material point P

It is assumed that the *motion* of this arbitrary material point in $\mathfrak{B}$ can be described

[1] The use of the upper case symbol for the position vector in κ_0 is an exception to the notation for vectors introduced above.

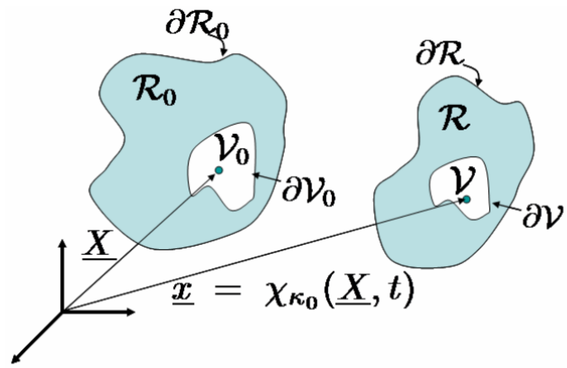

Figure 1.1: Schematic of notation used to identify material points and regions in an arbitrary body $\mathfrak{B}$ in the reference configuration κ_0 and current configuration $\kappa(t)$.

through a relationship of the form

$$\underline{x} \;=\; \underline{\chi}(\underline{X}, t) \qquad \text{for t} \geq 0. \tag{1.1}$$

With respect to the map given in (1.1), we will make the following *mathematical* assumptions:

1. The function $\underline{\chi}_{\kappa_0}(\underline{X}, t)$ is continuously differentiable in all variables, at least up through the second derivatives.
2. At each fixed time $t \geq 0$, the following property holds: for any $\underline{X}$ and corresponding $\underline{x}$, there are open balls $B_{\underline{X}}$ (centered at $\underline{X}$) and $B_{\underline{x}}$ (centered at $\underline{x}$), both contained in $\mathfrak{B}$, such that points of $B_{\underline{X}}$ are in one-to-one correspondence with points of $B_{\underline{x}}$.

These mathematical assumptions correspond to specific physical requirements. The first ensures a degree of smoothness of the trajectories of material points that will allow us to define quantities such as the velocity and acceleration. The second assumption implies the following two conditions are met:

- Impenetrability of the body
- No voids are created in the body.

The second condition also implies that the Jacobian J of the transformation (1.1) is non-zero at all times $t \geq 0$ and all points in the body. Here, J is

$$J \;=\; \det\left(\frac{\partial \underline{\chi}_{\kappa_0}(\underline{X}, t)}{\partial \underline{X}}\right). \tag{1.2}$$

If the reference configuration of the body was at any time t^* an actual configuration of the body, then at this time, $\underline{x} = \underline{X}$. At this time, it follows that $J(t^*) = 1$. However, as just stated, J can never vanish, so that if J is positive at one time, it must be positive at all times. We can therefore, conclude that under these conditions, for all points in the body,

$$J > 0, \text{for all } t \geq 0. \tag{1.3}$$

Furthermore, we can conclude that the transformation (1.1) posseses and inverse,

$$\underline{X} \;=\; \underline{\chi}^{-1}(\underline{x}, t). \tag{1.4}$$

For future discussions, it will be useful to introduce notation for identifying various regions of the body. The region occupied by the entire body in κ_0 will be denoted by $\mathcal{R}_0$ with closed boundary $\partial\mathcal{R}_0$, Figure 1.2. The corresponding regions and boundaries for the body in $\kappa(t)$ are $\mathcal{R}$ and $\partial\mathcal{R}$, respectively. An arbitrary material region within $\mathcal{B}$ in κ_0 will be denoted as $\mathcal{V}_0$ ($\mathcal{V}_0 \subseteq \mathcal{R}_0$) with boundary $\partial\mathcal{V}_0$. The corresponding subregion in the current configuration $\kappa(t)$ will be denoted as $\mathcal{V}$ ($\mathcal{V} \subseteq \mathcal{R}$) with boundary $\partial\mathcal{V}$.

For the current purposes, it suffices to define one reference configuration for the body. Hence, in further discussions, we drop the subscript κ_0 and it will be understood that the functions $\underline{\chi}(\underline{X}, t)$ and $\underline{\chi}^{-1}(\underline{x}, t)$ depend on the choice of reference configuration. However, in future discussion of multi-mechanism materials, more than one reference configuration will be employed. In these discussions, we will make use of subscripts of this kind to identify the appropriate reference configuration.

1.3 Referential and spatial descriptions

It follows from the second mathematical assumption that the vectors $\underline{X}$ and $\underline{x}$ for a material point P are in one-to-one correspondence. Therefore, field variables such as the density, ρ, can either be written as a function of $\underline{X}$ and t (the *referential* or *Lagrangian* description) or as a function of $\underline{x}$ and t, (the *spatial* or *Eulerian* description),

$$\rho \;=\; \bar{\rho}(\underline{X}, t) \;=\; \hat{\rho}(\underline{x}, t). \tag{1.5}$$

Clearly these two descriptions can be related using (1.1) and (1.4),

$$\hat{\rho}(\underline{x}, t) \;=\; \hat{\rho}(\underline{\chi}(\underline{X}, t), t) \equiv \bar{\rho}(\underline{X}, t) \tag{1.6}$$

and

$$\bar{\rho}(\underline{X}, t) \;=\; \bar{\rho}(\underline{\chi}^{-1}(\underline{x}, t), t) \equiv \hat{\rho}(\underline{x}, t) \tag{1.7}$$

The Eulerian description $\hat{\rho}(\underline{x}, t)$ is independent of information about the position of individual material particles and is typically used for the motion of viscous fluids.

When discussing entities such as the gradient, divergence or curl that involve spatial derivatives, we need to be clear whether we are considering these quantities in terms of position vector in the referential or current configuration. It will be convenient to use capital letters/lower case letters when we use the referential/current spatial variable. For example, for a field variable $f = \bar{f}(\underline{X}, t) = \hat{f}(\underline{x}, t)$,

$$\text{Grad}f \;=\; \frac{\partial \bar{f}(\underline{X}, t)}{\partial \underline{X}} \quad \text{and} \quad \text{grad}f \;=\; \frac{\partial \hat{f}(\underline{x}, t)}{\partial \underline{x}}. \tag{1.8}$$

1.4 Displacement, Velocity, and Acceleration

1.4.1 Displacement

The displacement $\underline{u}$ of a particle P at time t is defined by,

$$\underline{u} \;=\; \underline{x} \;-\; \underline{X}. \tag{1.9}$$

In writing (1.9), we have not specified how we are evaluating the right hand side of equation. In particular, if we make use of (1.1), we obtain the displacement in Eulerian form,

$$\hat{\underline{u}}(\underline{x}, t) \;=\; \underline{x} \;-\; \underline{\chi}^{-1}(\underline{x}, t). \tag{1.10}$$

Alternatively, using (1.4), we obtain the Lagrangian description,

$$\bar{\underline{u}}(\underline{X}, t) \;=\; \underline{\chi}(\underline{X}, t) \;-\; \underline{X}. \tag{1.11}$$

1.4.2 Velocity and Acceleration

The velocity $\underline{v}$ of a material point P is defined as the time derivative of the displacement (1.11) (for fixed material particle),

$$\underline{v} \;=\; \frac{\partial \bar{\underline{u}}(\underline{X}, t)}{\partial t} \quad \text{or} \quad \underline{v} \;=\; \frac{\partial \underline{\chi}(\underline{X}, t)}{\partial t}, \tag{1.12}$$

The acceleration of the same particle P is defined as the time derivative of the velocity, (for fixed material particle),

$$\underline{a} \;=\; \frac{\partial \underline{v}(\underline{X}, t)}{\partial t} \;=\; \frac{\partial^2 \underline{\chi}(\underline{X}, t)}{\partial t^2}. \tag{1.13}$$

As was done for the density field, both these field variables can be written in Eulerian form using (1.4). For example,

$$\underline{v} \;=\; \bar{\underline{v}}(\underline{X}, t) \;=\; \bar{\underline{v}}(\underline{\chi}^{-1}(\underline{x}, t), t) \;=\; \hat{\underline{v}}(\underline{x}, t). \tag{1.14}$$

Material Derivative

The material derivative of a field variable such as density, is defined as the partial derivative of the function with respect to time holding the material point fixed,

$$\frac{D\rho}{Dt} \;=\; \frac{\partial \bar{\rho}(\underline{X}, t)}{\partial t}. \tag{1.15}$$

Sometimes the material derivative is called the *total derivative* or *substantial derivative*. For example, it follows that the acceleration is the material derivative of the velocity field.

Frequently in fluid mechanics we work with the Eulerian representation of functions without knowledge of the deformation (1.1) with which to obtain the Lagrangian representation from the Eulerian representation. As a result, it is not possible to directly evaluate (1.15). Using the chain rule, the material derivative can be written for the spatial formulation of a field variable. For example, in the case of the density field,

$$\frac{D\rho}{Dt} = \frac{\partial \hat{\rho}(\underline{x},t)}{\partial t} + v_i \frac{\partial \hat{\rho}(\underline{x},t)}{\partial x_i} \qquad \text{or} \qquad \frac{D\rho}{Dt} = \frac{\partial \hat{\rho}(\underline{x},t)}{\partial t} + \underline{v}(\underline{x},t)\cdot \mathrm{grad}\rho(\underline{x},t). \tag{1.16}$$

Therefore, depending on the representation we have for a field variable, we have two choices for the material derivative. Once again, for a scalar field variable,

Material Derivative

$$\frac{D\rho}{Dt} = \begin{cases} \dfrac{\partial \bar{\rho}(\underline{X},t)}{\partial t} & \text{Lagrangian Representation} \\[2ex] \dfrac{\partial \hat{\rho}(\underline{x},t)}{\partial t} + \underline{v}(\underline{x},t)\cdot \mathrm{grad}\rho(\underline{x},t) & \text{Eulerian Representation} \end{cases} \tag{1.17}$$

Using (1.17), we may obtain the acceleration from the spatial representation of the velocity field,

$$\underline{a} = \frac{\partial \underline{v}(\underline{x},t)}{\partial t} + (\mathrm{grad}\underline{v}(\underline{x},t))\underline{v}(\underline{x},t) \qquad \text{or} \qquad a_i = \frac{\partial \hat{v}_i(\underline{x},t)}{\partial t} + \frac{\partial \hat{v}_i(\underline{x},t)}{\partial x_j} v_j. \tag{1.18}$$

In summary, we can calculate the acceleration from either the Lagrangian or Eulerian representation of the velocity vector using (1.13) or (1.18),

Acceleration

$$\underline{a} = \begin{cases} \dfrac{\partial \underline{v}(\underline{X},t)}{\partial t} \\[2ex] \dfrac{\partial \underline{v}(\underline{x},t)}{\partial t} + (\mathrm{grad}\underline{v}(\underline{x},t))\ \underline{v}(\underline{x},t) \end{cases} \tag{1.19}$$

1.5 Deformation Gradient of the Motion

Let $\underline{x}$ be the position of a material point P in κ and $\underline{y}$ be the position of another particle Q in the neighborhood $\mathcal{N}$ of P. We assume $\mathcal{N}$ is very small so that $\underline{y} - \underline{x}$ is an infinitesimal vector. The rectangular components of $dx_i = (dx_1, dx_2, dx_3)$. It can then be shown from (1.1), that

$$dx_i = \frac{\partial \chi_i(\underline{x},t)}{\partial X_A}\, dX_A. \tag{1.20}$$

It is straightforward to prove the components $\partial\chi_i/\partial X_A, (i, A = 1, 2, 3)$ transform under a change of basis like the components of a second order tensor. We denote this tensor as $\underline{F}$, so that,

Deformation Gradient Tensor:

$$\underline{F} \equiv \frac{\partial \underline{\chi}(\underline{X}, t)}{\partial \underline{X}} \qquad \text{or} \qquad F_{iA} = \frac{\partial \chi_i(\underline{X}, t)}{\partial X_A}.$$

and (1.21)

Inverse Deformation Gradient Tensor:

$$\underline{F}^{-1} \equiv \frac{\partial \underline{\chi}^{-1}(\underline{x}, t)}{\partial \underline{x}} \qquad \text{or} \qquad F^{-1}_{Ai} = \frac{\partial \chi^{-1}_A(\underline{x}, t)}{\partial x_i}.$$

Using this notation, it follows from (1.20),

$$\underline{dx} = \underline{F}\,\underline{dX}, \qquad \text{or} \qquad dx_i = F_{iA} dX_A. \tag{1.22}$$

It is clear from this last expression, that $\underline{F}$ determines how infinitesimal vectors in the

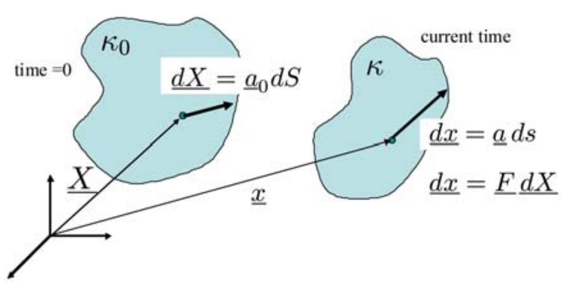

Figure 1.2: Deformation of an infinitesimal material element, $\underline{dX}$ in κ_0 which is transformed to $\underline{dx}$ in $\kappa(t)$.

reference configuration are deformed (stretched and rotated) during the motion, Fig. 1.3. For this reason, $\underline{F}$ is called the *deformation gradient tensor*. Using the notations introduced in (1.8) with (1.10) and (1.11), we see that $\underline{F}$ and $\underline{F}^{-1}$ are related to the gradients of the displacement vector through

$$\underline{F} = \text{Grad}\underline{u} + \underline{I}, \qquad \underline{F}^{-1} = \underline{I} - \text{grad}\underline{u}. \tag{1.23}$$

1.6 Transformations of infinitesimal areas and volumes during the deformation

The transformation of an infinitesimal material element $\underline{dX}$ from the reference configuration to the current configuration is determined through $\underline{F}$. This tensorial relationship can in turn be used to calculate the changes in material area and volume during the deformation.

1.6.1 Transformation of infinitesimal areas

If we consider two distinct infinitesimal material line elements $\underline{dY}, \underline{dZ}$ in configuration κ_0 at $\underline{X}$, then we may define an infinitesimal material area $\underline{dA}$ in κ_0 and the corresponding area $\underline{da}$ in $\kappa(t)$

$$\underline{dA} \equiv \underline{dY} \times \underline{dZ}, \qquad \underline{da} \equiv \underline{dy} \times \underline{dz}, \tag{1.24}$$

where $\underline{dA}$ can be written as the product of its magnitude dA and unit normal $\underline{N}$ such that $\underline{dA} = \underline{N} dA$. Similarly, $\underline{da} = \underline{n} da$ in $\kappa(t)$, where $\underline{n}$ is the unit normal and da is the magnitude of area $\underline{da}$. Noting from (1.22) that $\underline{dy} \equiv \underline{F}\, \underline{dY}$ and $\underline{dz} \equiv \underline{F}\, \underline{dZ}$, it then follows,

$$\begin{aligned} da_m &= \epsilon_{mjk}\, F_{jB}\, F_{kC}\, dY_B dZ_C \\ &= \epsilon_{ijk}\, \delta_{im}\, F_{jB}\, F_{kC}\, dY_B dZ_C \\ &= \epsilon_{ijk}\, F_{iA}\, F^{-1}_{Am}\, F_{jB}\, F_{kC}\, dY_B dZ_C \\ &= \epsilon_{ABC}\, J\, F^{-1}_{Am}\, dY_B dZ_C \\ &= J\, F^{-1}_{Am} dA_A \end{aligned} \tag{1.25}$$

Namely, we have Nanson's formula,

$$\boxed{\text{Nanson's Formula:} \qquad \underline{da} \;=\; J\underline{F}^{-T}\, \underline{dA}.} \tag{1.26}$$

1.6.2 Transformation of infinitesimal volumes

If we consider three distinct material non-planar line elements $\underline{dX}, \underline{dY}, \underline{dZ}$ in configuration κ_0 at $\underline{X}$, then we may define an infinitesimal material volume dV in κ_0 with volume $\underline{dv}$ in $\kappa(t)$ as,

$$dV \equiv \underline{dX} \cdot (\underline{dY} \times \underline{dZ}), \qquad dv \equiv \underline{dx} \cdot (\underline{dy} \times \underline{dz}), \tag{1.27}$$

where $\underline{dx} \equiv \underline{F}\, \underline{dX}$, $\underline{dy} \equiv \underline{F}\, \underline{dY}$ and $\underline{dz} \equiv \underline{F}\, \underline{dZ}$. It then follows that,

$$\begin{aligned} dv &= \epsilon_{ijk}\, F_{iA}\, F_{jB}\, F_{kC}\, dX_A\, dY_B\, dZ_C \\ &= \epsilon_{ABC}\, J dX_A\, dY_B\, dZ_C \end{aligned} \tag{1.28}$$

and therefore,

$$\boxed{dv \;=\; J\, dV.} \tag{1.29}$$

Note, that in obtaining (1.26) and (1.28), we have made use of the following result for a second order tensor, $\underline{A}$,

$$\epsilon_{ijk}\det A \;=\; \epsilon_{lmn}\, A_{li}\, A_{mj}\, A_{nk}. \tag{1.30}$$

It will be useful in the discussion of the governing equations to note the following result for the material derivative of the Jacobian of the transformation (1.1),

$$\frac{DJ}{Dt} \;=\; J\,\mathrm{div}\,\underline{v}, \tag{1.31}$$

where $\underline{v}$ is the velocity vector.

1.7 Measures of Stretch and Strain

In the last section, we discussed the deformation gradient tensor, which completely determines the mapping of an infinitesimal material element from the reference to current configuration. In this section, we consider specific characteristics of this mapping. In particular, we consider the stretch and strain undergone by this material element. Unlike the deformation gradient, which has a unique definition, the strain of an infinitesimal material element can be defined in a number of ways.

1.7.1 Material stretch and strain tensors, $\underline{C}$ and $\underline{E}$

We first consider an infinitesimal material element $\underline{dX} = \underline{a}_0 dS$ in the κ_0 which is mapped to $\underline{dx} = \underline{a}$ ds in $\kappa(t)$ where $\underline{a}_0$ and $\underline{a}$ are unit vectors, Figure 1.3. The stretch undergone by this material element during the deformation is defined as $\lambda \equiv |\underline{dx}|/|\underline{dX}| = ds/dS$. Using (1.22), we can rewrite magnitude of the infinitesimal vector $\underline{dx}$ as follows,

$$ds^2 \;=\; |\underline{dx}|^2 = \underline{dx}\cdot\underline{dx} = \underline{a}_0\cdot(\underline{F}^T\underline{F}\,\underline{a}_0)dS^2 \tag{1.32}$$

Recalling the definition of the right Cauchy-Green tensor (sometimes called the Green deformation tensor)

$$\boxed{\textbf{Right Cauchy-Green Tensor:}\qquad \underline{C}\equiv\underline{F}^T\underline{F},\qquad \text{or}\qquad C_{AB}\equiv F_{iA}\,F_{iB}}, \tag{1.33}$$

we therefore have from (1.32),

$$\boxed{\lambda^2 \;=\; \underline{a}_0\cdot\underline{C}\,\underline{a}_0.} \tag{1.34}$$

Namely, the stretch of a material element located at $\underline{X}$ in κ_0, can be calculated at arbitrary time t solely from knowledge of its orientation in κ_0 and the tensor $\underline{C}$ at $\underline{X}$ at time t.

It can be shown that, for all material points in the body at all times, $\underline{C}$ is symmetric and positive definite

$$\underline{u}\cdot(\underline{C}\,\underline{u})>0 \qquad \text{for all}\;\; \underline{u}\neq 0. \tag{1.35}$$

The proof of symmetry follows directly from the definition (1.33)

$$\underline{C}^T \;=\; (\underline{F}^T\underline{F})^T \;=\; \underline{F}^T\underline{F} \;=\; \underline{C}, \tag{1.36}$$

Physical Significance of Diagonal Elements of $\underline{C}$

Consider at infinitesimal element represented by $\underline{dX}$ in the reference configuration and $\underline{dx}$ in the current configuration. If we now consider the special case where $\underline{dX} = dS\underline{e}_1$, then it follows from (1.34), that

$$\frac{ds^2}{dS^2} \; = \; C_{11}. \tag{1.37}$$

The ratio ds^2/dS^2 is typically called the stretch ratio and denoted by λ. We can describe the result (1.37) in words as,

> C_{11} is equal to the stretch ratio squared of an infinitesimal material element was aligned with the $\underline{e}_1$ axis in the reference configuration.

More generally, the diagonal elements of $\underline{C}$ have a specific physical meaning. C_{ii} (no sum on i) is the square of the stretch of a material element which was aligned with the direction $\underline{e}_i$ in κ_0.

Physical significance of off-diagonal elements of $\underline{C}$

Now consider two infinitesimal material elements corresponding to $\underline{dX}^{(1)} = dS^{(1)}\underline{a}_0^{(1)}$ and $\underline{dX}^{(2)} = dS^{(2)}\underline{a}_0^{(2)}$. where $\underline{a}_0^{(1)}$ and $\underline{a}_0^{(2)}$ are unit vectors. In the current configurations, these infinitesimal elements are denoted as $\underline{dx}^{(1)} = ds^{(1)}\underline{a}^{(1)}$ and $\underline{dx}^{(2)} = ds^{(2)}\underline{a}^{(2)}$, respectively. The angle between these same material elements in the current configuration will be denoted as α. Therefore,

$$\cos\alpha \; = \; \frac{\underline{dx} \cdot \underline{dx}}{ds^{(1)}ds^{(2)}} \; = \; \frac{F_{iA}F_{iB}a_{0_A}^{(1)}a_{0_B}^{(1)}dS^{(1)}dS^{(2)}}{ds^{(1)}ds^{(2)}} \; = \; \frac{C_{AB}a_{0_A}^{(1)}a_{0_B}^{(1)}}{\lambda^{(1)}\,\lambda^{(2)}}. \tag{1.38}$$

Now consider the special case, where the infinitesimal elements are perpendicular in the reference configuration, for example, $\underline{a}_0^{(1)} = \underline{e}_1$ and $\underline{a}_0^{(2)} = \underline{e}_2$, in which case, (1.38) reduces to,

$$\cos\alpha \; = \; \frac{C_{12}}{\sqrt{C_{11}C_{22}}}. \tag{1.39}$$

For example, if the angle between the elements remains unchanged (still 90^o), then $C_{12} = 0$. If the angle decreases from 90^o, then C_{12} will be a positive number. If the $\alpha \in (90^o, 270^o)$, then C_{12} will be negative.

Lagrangian Strain

In addition, to considering the stretch of an infinitesimal material element, it is of interest to study its strain. While there is no unique definition, one measure of interest is the change in the square of the magnitude of the infinitesimal material element, relative to its magnitude squared in κ_0. This strain measure can be related to the right Cauchy-Green tensor using (1.34),

$$\frac{ds^2 - dS^2}{dS^2} \;=\; (\lambda^2 - 1) \;=\; \underline{a}_0 \cdot (\underline{C} - \underline{I})\, \underline{a}_0. \tag{1.40}$$

The quantity in brackets is twice the Green-Lagrange Strain Tensor

$$\boxed{\textbf{Green-Lagrange Strain Tensor:} \qquad \underline{E} \equiv \frac{1}{2}(\underline{C} - \underline{I}).} \tag{1.41}$$

The motivation for the factor of one-half in this definition will become clear when linear elasticity is considered. It therefore follows from (1.40) and (1.41) that,

$$\boxed{\frac{1}{2}\frac{ds^2 - dS^2}{dS^2} \;=\; \underline{a}_0 \cdot \underline{E}\, \underline{a}_0.} \tag{1.42}$$

From (1.41), we see that the diagonal components of $\underline{E}$ have physical significance. E_{ii} (no sum on i) is the strain undergone by a material element which is tangent to $\underline{e}_i$ in κ_0. The strain is taken relative to dS, the length of this element in κ_0. Using the definition of the strain tensors with (1.23) and (1.33), we can write the Green-Lagrange strain with respect to the displacement,

$$\underline{E} \;=\; \frac{1}{2}\left(\text{Grad}\underline{u} \;+\; \text{Grad}\underline{u}^T \;+\; \text{Grad}\underline{u}^T\, \text{Grad}\underline{u}\right) \tag{1.43}$$

or

$$E_{AB} \;=\; \frac{1}{2}\left(\frac{\partial u_A}{\partial X_B} \;+\; \frac{\partial u_B}{\partial X_A} \;+\; \frac{\partial u_C}{\partial X_A}\frac{\partial u_C}{\partial X_B}\right). \tag{1.44}$$

1.7.2 Spatial Stretch and Strain Tensors, $\underline{b}, \underline{e}$

Alternatively, we can consider the stretch and strain of material elements, calculated solely from knowledge of their direction in $\kappa(t)$ and appropriate kinematic tensors. For example, in this case, we consider a strain measure relative to $\kappa(t)$,

$$\frac{ds^2 - dS^2}{ds^2}, \tag{1.45}$$

and note that dS^2 can be written in terms of ds and $\underline{a}$, defined in $\kappa(t)$,

$$dS^2 \;=\; |\underline{dX}|^2 \;=\; \underline{dX} \cdot \underline{dX} \;=\; \underline{a} \cdot (\underline{F}^{-T}\underline{F}^{-1}\, \underline{a})ds^2 \;=\; \underline{a} \cdot (\underline{b}^{-1}\underline{a})ds^2 \tag{1.46}$$

where $\underline{b}$ is the left Cauchy-Green tensor (sometimes called the Finger deformation tensor)

$$\boxed{\textbf{Left Cauchy-Green Tensor:} \qquad \underline{b} \equiv \underline{F}\,\underline{F}^T, \qquad \text{or} \qquad b_{ij} \equiv F_{iA}\,F_{jA}}, \tag{1.47}$$

The left Cauchy-Green tensor is also symmetric and positive definite. Upon substituting (1.46) in (1.45), we find

$$\boxed{\frac{1}{2}\frac{ds^2 - dS^2}{ds^2} \;=\; \frac{1}{2}\underline{a}\cdot(\underline{I} - \underline{b}^{-1})\,\underline{a} \;=\; \underline{a}\cdot\underline{e}\,\underline{a},} \tag{1.48}$$

where,we have introduced a spatial strain tensor, $\underline{e}$,

$$\boxed{\textbf{Euler-Almansi Strain Tensor:} \qquad \underline{e} \equiv \frac{1}{2}(\underline{I} - \underline{b}^{-1})} \tag{1.49}$$

referred to as the *Euler-Almansi Strain Tensor*. From (1.48), we see each diagonal component $\underline{e}_{ii}$ (no sum on i) is the strain undergone by a material element which is tangent to $\underline{e}_i$ in κ. The strain is taken relative to the length in κ. Furthermore, using (1.46), we obtain

$$\boxed{\frac{1}{\lambda^2} \;=\; \underline{a}\cdot\underline{b}^{-1}\,\underline{a}.} \tag{1.50}$$

enabling the stretch to be calculated for an infinitesimal material element, solely from its direction in κ and knowledge of $\underline{b}^{-1}$ at the same material point and time. Using the (1.49) with (1.23) and (1.47), we can write the Euler-Almansi Strain Tensor with respect to the gradient of the displacement,

$$\underline{e} \;=\; \frac{1}{2}\left(\text{grad}\underline{u} \;+\; \text{grad}\underline{u}^T \;-\; \text{grad}\underline{u}^T\,\text{grad}\underline{u}\right) \tag{1.51}$$

or

$$e_{ij} \;=\; \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} \;+\; \frac{\partial u_j}{\partial x_i} \;-\; \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right). \tag{1.52}$$

Because $\underline{b}$ and $\underline{e}$ are used to calculate the stretch and strain of material elements based on knowledge of the orientation of these material elements in $\kappa(t)$, they are often referred to as spatial (or Eulerian) measures of stretch and strain, respectively.

1.8 Polar Decomposition of the Deformation Gradient Tensor

Recall the **Polar Decomposition Theorem:**

An arbitrary non-singular tensor $\underline{F}$ can be represented as

$$\underline{F} \;=\; \underline{R}\,\underline{U} \;=\; \underline{V}\,\underline{R} \qquad \text{or} \qquad F_{iA} \;=\; R_{iB}U_{BA} \;=\; V_{ij}R_{jA} \tag{1.53}$$

where $\underline{U}$ and $\underline{V}$ are positive definite symmetric tensors and $\underline{R}$ is an orthogonal tensor. Futhermore, the right and left decompositions given in (1.53) are unique.

When this theorem is applied to the deformation gradient tensor, necessarily $\underline{R}$ is proper orthogonal (since $J > 0$) and the tensor $\underline{R}$ is sometimes called the *rotation tensor.* The tensors $\underline{U}$ and $\underline{V}$ are referred to respectively as the *right and left stretch tensors*. It follows from (1.33),(1.47) and (1.53) that

$$\underline{C} = \underline{U}^2 \qquad \underline{b} = \underline{V}^2. \tag{1.54}$$

Noting that,

$$\underline{dx} = \underline{F}\,\underline{dX} = \underline{R}\,(\underline{U}\,\underline{dX}). \tag{1.55}$$

If we define $\underline{dX}' \equiv \underline{U}\,\underline{dX}$, then,

$$\underline{dx} = \underline{R}\,\underline{dX}'. \tag{1.56}$$

and therefore,

$$|\underline{dx}| = R_{ij}dX'_jR_{ik}dX'_k = \delta_{jk}dX'_jdX'_k = |\underline{dX}'|. \tag{1.57}$$

It is clear that $\underline{R}$ does not contribute to the stretch of the infinitesimal material element, only to the rotation. It follows from (1.53), that $\underline{U}$ and $\underline{V}$ are related through,

$$\underline{U} = \underline{R}^T\underline{V}\,\underline{R}. \tag{1.58}$$

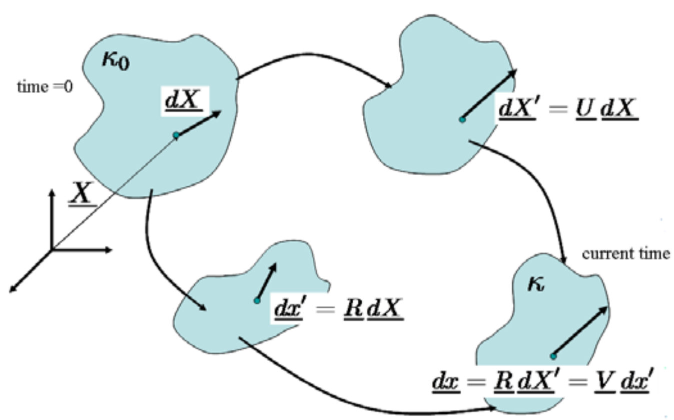

Figure 1.3: Schematic of Polar Decomposition of the Deformation, through $\underline{F} = \underline{RU} = \underline{V}\,\underline{R}$.

Principal values and Principal vectors of $\underline{U}, \underline{V}$

Since $\underline{U}, \underline{V}$ are symmetric tensors, they each possess three principal values and an orthonormal set of associated principal directions. The principal vectors of $\underline{U}$ and $\underline{V}$ are referred to respectively as the *right and left principal directions of stretch* and denoted as $\underline{u}^i$, $\underline{v}^i$. It is straightforward to show that the principal vectors are related through,

$$\underline{v}^i = \underline{R}\,\underline{u}^i. \tag{1.59}$$

and the principal values of $\underline{U}$ and $\underline{V}$ are identical. Therefore, $\underline{U}$ and $\underline{V}$ can be represented as (sum on i),

$$\underline{U} \;=\; \lambda_i \underline{u}^i \otimes \underline{u}^i \qquad \underline{V} \;=\; \lambda_i \underline{v}^i \otimes \underline{v}^i. \tag{1.60}$$

It then follows that (sum on i),

$$\underline{F} \;=\; \lambda_i \underline{v}^i \otimes \underline{u}^i, \qquad \underline{R} \;=\; \underline{v}^i \otimes \underline{u}^i, \qquad \underline{C} \;=\; \lambda_i^2 \underline{u}^i \otimes \underline{u}^i, \qquad \underline{B} \;=\; \lambda_i^2 \underline{v}^i \otimes \underline{v}^i. \tag{1.61}$$

1.9 Other Strain Measures

As discussed earlier, the choice of stretch or strain measure is not unique. One alternative is the Hencky strain [2].

$$\begin{array}{ll} \textbf{HenckyStrain}: & \\ \text{Material Form:} & \ln \underline{U} \\ \text{Spatial Form:} & \ln \underline{V} \end{array} \tag{1.62}$$

Another choice is the Biot strain,

$$\begin{array}{ll} \textbf{BiotStrain}: & \\ \text{Material Form:} & \underline{U} - \underline{I} \\ \text{Spatial Form:} & \underline{V} - \underline{I} \end{array} \tag{1.63}$$

More generally, we can define,

$$(\underline{U}^n - \underline{I}), \qquad (\underline{V}^n - \underline{I}), \tag{1.64}$$

where $n = 2$ and $n = -2$ we recover the Euler-Almansi and the Green-Lagrange strains discussed earlier.

1.10 Velocity Gradient

We denote $\underline{L}$ as the gradient of the spatial form of the velocity vector, so that the components of $\underline{L}$ with respect to rectangular coordinates are

$$L_{ij} \;=\; \frac{\partial v_i(\underline{x}, t)}{\partial x_j}. \tag{1.65}$$

Recall that any second order tensor can be decomposed into the sum of a symmetric and skew symmetric second order tensor. We can represent $\underline{L}$ in this way,

$$\text{Ł}_{ij} \;=\; \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \;+\; \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right). \tag{1.66}$$

[2] Hencky, H.: Uber die Form des Elastizitatsgesetzes bei ideal elastischen Stoffen. Zeit. Tech. Phys., vol. 9, 1928, pp. 215-220, 457

We denote the symmetric part of $\underline{L}$ by $\underline{D}$:

Rate of Deformation Tensor *or* **Rate of Strain Tensor**	$\underline{D} = \frac{1}{2}\left(\text{grad}\underline{L} + \text{grad}^T\underline{L}\right).$

and the skew-symmetric part of $\underline{L}$ by $\underline{W}$:

Spin Tensor *or* **Vorticity Tensor**	$\underline{W} = \frac{1}{2}\left(\text{grad}\underline{L} - \text{grad}^T\underline{L}\right).$

In indicial notation,

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right). \tag{1.67}$$

Physical Significance of Diagonal Elements of $\underline{D}$

The physical significance of $\underline{D}$ can be studied by considering the rate of change in magnitude of an infinitesimal material element $\underline{dx}$ of length ds. We first consider the rate of chance of ds^2 which is equal to

$$\frac{D(ds^2)}{Dt} = 2\,dx_i \frac{Ddx_i}{Dt}. \tag{1.68}$$

The rate of change in the infinitesimal material element $\underline{dx}$ is,

$$\begin{aligned} \frac{Ddx_i}{Dt} &= \frac{Ddx_i}{Dt} \\ &= \frac{\partial F_{iA}}{\partial t}\,dX_A \\ &= \frac{\partial^2 \chi_i(\underline{X}, t)}{\partial t \partial\, X_A}\,dX_A \\ &= \frac{\partial v_i}{\partial x_j}\,F_{jA}\,dX_A \\ &= L_{ij} dx_j \end{aligned} \tag{1.69}$$

and so from (1.68) and (1.69) we find,

$$\frac{D(ds^2)}{Dt} = 2\,\underline{dx} \cdot (\underline{Ddx}). \tag{1.70}$$

and thus,

$$\frac{Dds}{Dt} = \frac{\underline{dx} \cdot (\underline{D}\underline{dx})}{ds}. \tag{1.71}$$

We can interpret the meaning of each of the diagonal elements of $\underline{D}$ by judicious choice of $\underline{dx}$. For example, choosing $\underline{dx} = ds\underline{e}_1$ at time t, we find that

$$\frac{Dds}{Dt} = D_{11} ds. \tag{1.72}$$

Namely, D_{11} is the rate of change of ds divided by ds of a material element which at time t is aligned with the $\underline{e}_1$ axis. So, for a physical flow, if we expect that such a material element does not change length at time t, then $D_{11} = 0$. Alternatively, if we expect that it is getting longer, then $D_{11} > 0$. The other diagonal elements can be interpreted in a similar way.

Physical significance of off-diagonal elements of $\underline{D}$

Now consider two infinitesimal material elements $\underline{x}^{(1)}$ and $\underline{x}^{(2)}$ which intersect at angle α with lengths $|\underline{dx}|$ and $|\underline{dy}|$ respectively. Then,

$$\cos\alpha = \frac{\underline{dx}^{(1)} \cdot \underline{dx}^{(2)}}{|\underline{dx}^{(1)}|\,|\underline{dx}^{(2)}|} \tag{1.73}$$

and therefore,

$$\begin{aligned}
\frac{D}{Dt}\cos\alpha &= \frac{D}{Dt}\left(\frac{\underline{dx}^{(2)} \cdot \underline{dx}^{(2)}}{|\underline{dx}^{(1)}|\,|\underline{dx}^{(2)}|}\right) \\
&= \frac{D}{Dt}\left(\underline{dx}^{(1)} \cdot \underline{dx}^{(2)}\right)\frac{1}{|\underline{dx}^{(1)}|\,|\underline{dx}^{(2)}|} - \frac{\underline{dx}^{(1)} \cdot \underline{dx}^{(1)}}{|\underline{dx}^{(1)}|^2\,|\underline{dx}^{(2)}|^2}\frac{D}{Dt}\left(|\underline{dx}^{(1)}|\,|\underline{dx}^{(2)}|\right) \\
&= 2\,\frac{\underline{dx}^{(2)} \cdot (\underline{D}\underline{dx}^{(1)})}{|\underline{dx}^{(1)}|\,|\underline{dx}|} - \frac{\underline{dx}^{(1)} \cdot \underline{dx}^{(2)}}{|\underline{dx}^{(1)}|^2\,|\underline{dx}^{(2)}|^2}\frac{D}{Dt}\left(|\underline{dx}^{(1)}|\,|\underline{dx}^{(2)}|\right)
\end{aligned} \tag{1.74}$$

For example, if are interested in the physical meaning of D_{12}, we can consider (1.74) for two infinitesimal material elements, $\underline{dx}^{(1)}, \underline{dx}^{(2)}$ which at time t are parallel to the base vectors $\underline{e}_1$ and $\underline{e}_2$. Since they are orthogonal, the second term in (1.74) drops out and we can show, the rate of change of the angle between these two vectors at time t is

$$\frac{D\alpha}{Dt} = -\,2\,D_{12}. \tag{1.75}$$

Therefore, the value of $D_{12}(\underline{x}, t)$ is the rate of change of angle between the two infinitesimal vectors which at time t are located at position $\underline{x}$ and parallel to base vectors, $\underline{e}_1$ and $\underline{e}_2$. Similar arguments holds for the other off diagonal elements. Notice that this interpretation of the components of $\underline{D}$ does not require knowledge of the behavior of specific material elements. Rather $\underline{D}(\underline{x}, t)$ is related to the rate of change of material elements which at time t are located at position $\underline{x}$.

1.10.1 Vorticity Vector and Vorticity Tensor

Recall the definition of the vorticity, $\underline{\omega}$ of the velocity vector[3],

$$\underline{\omega} \equiv \text{curl}\underline{v}, \qquad \text{or} \qquad \omega_i \equiv \epsilon_{ijk} \frac{\partial v_k}{\partial x_j}. \tag{1.76}$$

It is a nice exercise to show that the components of the vorticity vector are related to the components of the vorticity tensor through,

$$\omega_i \;=\; -\,\epsilon_{ijk}\, W_{jk}, \qquad \text{and} \qquad W_{ij} \;=\; -\frac{1}{2}\epsilon_{ijk}\,\omega_k. \tag{1.77}$$

1.11 Special Motions

1.11.1 Isochoric Motions

Isochoric motions are those in which the volume occupied by any set of fixed material particles is unchanged during the motion. Namely, $dv = dV$ for isochoric motions at all points in the body. A material does not have to be incompressible to undergo isochoric motions. However, *all* motions experienced by incompressible materials must be isochoric. We see from (1.28) that if a motion is isochoric then the value of J is one throughout the motion. In this case, we have from (1.31) that the divergence of $\underline{v}$ is equal to zero and hence the trace of $\underline{D}$ is zero. In summary, each of the following are necessary and sufficient conditions for a motion to be isochoric,

Necessary and sufficient conditions for an isochoric motion

$dv \;=\; dV, \quad J = 1, \quad DJ/Dt \;=\; 0, \quad \text{div}\,\underline{v} \;=\; 0, \quad \text{trace}\underline{D} \;=\; 0,$

at all points in the body, for all times during the motion.

1.11.2 Irrotational Motions

Irrotational motions are those for which the vorticity is zero. From (1.77), the vorticity tensor $\underline{W}$ is zero for irrotational motions.

1.11.3 Homogeneous Deformations

A motion is said to be a *homogeneous deformation* if $\underline{F}$ is independent of the material point. In this case, the can integrate (1.21) with respect to $\underline{X}$ to obtain,

$$\underline{x} \;=\; \underline{F}(t)\underline{X} \;+\; \underline{c}(t). \tag{1.78}$$

[3] Note that alternate definitions of vorticity are sometimes used. For example, sometimes the vorticity is defined as the negative of that given in (1.76). In other cases, the vorticity is taken to be twice that in (1.76).

It then follows directly that, $\underline{R}, \underline{U}, \underline{B}, \underline{C}, \underline{E}, \underline{e}$ and J are constant throughout the body (and are also only functions of time). Hence, in a homogeneous deformation, all material elements which are parallel, experience the same change in length and orientation. In particular, a straight material line in the reference configuration will remain straight throughout the deformation though it may change orientation.

As discussed below, two special categories of homogeneous deformations are *Rigid Motions* in which ($\underline{U} = \underline{I}$) and *Pure Homogeneous Deformations* in which ($\underline{R} = \underline{I}$).

1.11.4 Rigid Motions

We shall say that body $\mathfrak{B}$ performs a rigid motion if and only if the distance between two arbitrary material points remains the same for all time during the motion. Equivalently we can say, the for all arbitrary $\underline{X}, \underline{Y}$ in $\mathfrak{B}$,

$$|\underline{x} - \underline{y}| \; = \; |\underline{X} - \underline{Y}|, \tag{1.79}$$

for all times during the motion, where $\underline{x} = \underline{\chi}(\underline{X}, t)$ and $\underline{y} = \underline{\chi}(\underline{Y}, t)$. It can be shown (homework) that the most general rigid motion can be written as,

$$\underline{x} \; = \; \underline{F}(t)\underline{X} \; + \; \underline{c}(t), \qquad \text{with} \quad \underline{F}\underline{F}^T \; = \; \underline{I}. \tag{1.80}$$

where it should be recalled from (1.3) that the $\det\underline{F} > 0$. Namely, a rigid body motion is a homogeneous deformation for which $\underline{F}$ is proper orthogonal. Alternatively, we can write,

$$\underline{x} - \underline{y} \; = \; \underline{F}(t)(\underline{X} - \underline{Y}), \qquad \text{with} \quad \underline{F}\underline{F}^T \; = \; \underline{I}. \tag{1.81}$$

It can be shown (homework), that a necessary and sufficient condition for a motion to be rigid is that $\underline{F} = \underline{R}$ for all points in $\mathfrak{B}$ at all times during the rigid motion. Similarly, we can show,

$$\begin{array}{llll} \underline{C} = \underline{I}, & \underline{b} = \underline{I}, & \underline{U} = \underline{I} & \underline{V} = \underline{I}, \\ \underline{E} = \underline{0}, & \underline{e} = \underline{0}, & ds = dS & \lambda = 1. \end{array} \tag{1.82}$$

In addition, we see from (1.71) that for rigid motions, $\underline{D}$ must be identically zero at all points in the body for all time during the motion. Alternatively, we see from (1.71) that motions in which $\underline{D}$ is zero at all points in the body for all time are rigid motions. Namely, a motion of $\mathfrak{B}$ is rigid if and only if $\underline{D}$ is identically zero for all points in $\mathfrak{B}$ throughout the duration of the motion and therefore, $\underline{L} = \underline{W}$.

Velocity field for a rigid motion

Taking the material derivative of (3.66), we find,

$$\underline{v}(\underline{x}, t) - \underline{v}(\underline{y}, t) \; = \; \dot{\underline{F}}(t)(\underline{X} - \underline{Y}). \tag{1.83}$$

It then follows directly from (3.66) that $(\underline{X} - \underline{Y}) = \underline{F}^T(\underline{x} - \underline{y})$ so that,

$$\underline{v}(\underline{x}, t) - \underline{v}(\underline{y}, t) \; = \; \dot{\underline{F}}\underline{F}^T(\underline{x} - \underline{y}). \tag{1.84}$$

Then, recalling that in general $\dot{\underline{F}} = \underline{L}\underline{F}$ and for a rigid motion $\underline{L} = \underline{W}$,

$$\underline{v}(\underline{x}, t) - \underline{v}(\underline{y}, t) \; = \; \underline{W}(\underline{x} - \underline{y}). \tag{1.85}$$

It then follows from (1.77),

$$\underline{v}(\underline{x}, t) - \underline{v}(\underline{y}, t) \; = \; \frac{1}{2}\omega \times (\underline{x} - \underline{y}). \tag{1.86}$$

1.11.5 Pure Homogeneous Deformation

Pure homogeneous deformations are homogeneous deformations with $\underline{R} = \underline{I}$. Namely, $\underline{F} = \underline{U}(t) = \underline{V}(t)$, so

$$\underline{x} \; = \; \underline{U}(t)\underline{X} \; + \; \underline{c}(t). \tag{1.87}$$

We will consider two special types of Pure Homogeneous Deformations: Plane Strain and Axisymmetric Strain.

1.11.6 Plane Strain

In Plane Strain, the deformation is confined to one plane and $\underline{U}$ and the eigenvalues of $\underline{U}$ are of the form $(\lambda, \mu, 1)$. Namely, with respect to a basis composed of the principal directions of $\underline{U}$, the components of the $\underline{F}$ can be written as,

$$[\underline{F}] \; = \; [\underline{U}] \; = \; \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.88}$$

or, in shorthand, we will write:

$$\mathrm{diag}(\underline{F}) \; = \; \mathrm{diag}(\underline{U}) \; = \; (\lambda, \mu, 1). \tag{1.89}$$

It is clear that for the deformation given in (1.96) that material lines elements parallel to $\underline{u}_3$ direction do not undergo any stretching or rotation, hence the name plane strain. We will consider three special cases of plane strain.

Uniaxial strain
Uniaxial strain is a particular case of plane strain with

$$\mathrm{diag}(\underline{F}) \; = \; \mathrm{diag}(\underline{U}) \; = \; (\lambda, 1, 1). \tag{1.90}$$

Axisymmetric Plane Strain

An important special case of plane strain is axisymmetric plane strain in which

$$\mathrm{diag}(\underline{F}) \;=\; \mathrm{diag}(\underline{U}) \;=\; (\lambda, \lambda, 1). \tag{1.91}$$

Isochoric Plane Strain (or "Pure Shear")

Isochoric plane strain is a special case of plane strain in which the determinant of the deformation gradient tensor must be one, and so,

$$\mathrm{diag}(\underline{F}) \;=\; \mathrm{diag}(\underline{U}) \;=\; (\lambda, 1/\lambda, 1). \tag{1.92}$$

1.11.7 Axisymmetric Strain

Axisymmetric strain is a pure homogeneous deformation for which there exists an orthonormal basis such that,

$$\mathrm{diag}(\underline{F}) \;=\; \mathrm{diag}(\underline{U}) \;=\; (\mu, \mu, \lambda). \tag{1.93}$$

Two types of plane strain considered above, are also types of axisymmetric strain: uniaxial strain and axisymmetric plane strain. We will consider two additional types of axisymmetric strain: pure dilation and isochoric axisymmetric strain.

Pure Dilation

A pure dilation is a special pure axisymmetric homogeneous deformation for which

$$\underline{U} \;=\; \lambda \underline{I} \tag{1.94}$$

where λ are the principal stretches of $\underline{U}$. It follows from, (1.32) that,

$$\underline{dx} \;=\; \lambda\, \underline{dX}_o, \qquad \text{and} \qquad ds^2 \;=\; \lambda^2\, dS^2 \tag{1.95}$$

for all infinitesimal vectors $\underline{dX}$. Notice that infinitesimal elements do not change direction, during the deformation, they only undergo stretch. In addition, every material element undergoes the same stretch. As a result, a spherical volume will remain spherical and more generally, bodies will remain geometrically similar with a change of scale λ. The only isochoric pure dilation is one for which $\lambda = 1$.

Isochoric Axisymmetric Strain

In isochoric axisymmetric strain, the determinant of the deformation tensor must be identically one and hence,

$$\mathrm{diag}(\underline{F}) \;=\; \mathrm{diag}(\underline{U}) \;=\; (1/\sqrt{\lambda}, 1/\sqrt{\lambda}, \lambda). \tag{1.96}$$

Homogeneous Deformations: $\nabla \underline{F} = 0$	
Rigid Motions	$\underline{F} = \underline{R}$
Pure Homogeneous Deformations	$\text{diag}(\underline{F}) = \text{diag}(\underline{U}) = (\lambda, \mu, \gamma)$
Plane Strain	$\text{diag}(\underline{U}) = (\lambda, \mu, 1)$
Uniaxial Strain	$\text{diag}(\underline{U}) = (\lambda, 1, 1)$
Axisymmetric Plane Strain	$\text{diag}(\underline{U}) = (\lambda, \lambda, 1)$
Isochoric Plane Strain (”Pure Shear”)	$\text{diag}(\underline{U}) = (1/\lambda, \lambda, 1)$
Axisymmetric Strain	$\text{diag}(\underline{U}) = (\mu, \mu, \lambda)$
Uniaxial Strain	$\text{diag}(\underline{U}) = (1, 1, \lambda)$
Axisymmetric Plane Strain	$\text{diag}(\underline{U}) = (\mu, \mu, 1)$
Pure Dilation	$\text{diag}(\underline{U}) = (\lambda, \lambda, \lambda)$
Isochoric Axisymmetric Strain	$\text{diag}(\underline{U}) = (1/\sqrt{\lambda}, 1/\sqrt{\lambda}, \lambda)$

with a discussion of one-parameter deformations.

Other Homogeneous Deformations

Notice that there are an infinite number of deformations which, under a polar decomposition, have the same $\underline{U}$ as these pure homogeneous deformations, but for which $\underline{R}$, though constant is not equal to the identity tensor. Once simple way to check this is to compare the principal invariants of $\underline{C}$. Consider the case of simple shear.

Simple Shear

In simple shear, the deformation can be described relative to Cartesian coordinates through,

$$\underline{x} = \underline{X} + \kappa(t) \underline{X}_2 \underline{e}_1. \tag{1.97}$$

A rectangular block with sides parallel to the coordinate axes is transformed in simple shear to a skewed parallel piped. Material planes parallel to the X-axis remain parallel to the axis, though shifted by $\kappa(t) X_2$ in the X_1-direction (the *direction of shear*). These material planes are called glide planes and $\kappa(t)$ is called the *amount of shear*. Material planes parallel to the X_2-axis are rotated through an angle γ, the *angle of shear*, where $\tan\gamma = K$. The $X_1 - X_2$ plane is called the *plane of shear*. The components of the corresponding deformation gradient tensor,

$$[\underline{F}] = \begin{bmatrix} 1 & \kappa(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.98}$$

and the components of the left and right Cauchy-Green strain tensors are

$$[\underline{B}] = \begin{bmatrix} 1+\kappa(t)^2 & \kappa(t) & 0 \\ \kappa(t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [\underline{C}] = \begin{bmatrix} 1 & \kappa(t) & 0 \\ \kappa(t) & 1+\kappa(t)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.99}$$

As can be seen in (1.99), simple shear is a homogeneous, planar isochoric deformation. Noting, that in a homogeneous plane deformation, without loss in generality the rotation tensor can be represnted as,

$$[\underline{R}] = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.100}$$

It then follows from (1.99) and (1.100) that,

$$[\underline{U} = \underline{R}^T \underline{F}] = \begin{bmatrix} \cos\omega & \kappa(t)\cos\omega + \sin\omega & 0 \\ -\sin\omega & -\kappa(t)\sin\omega + \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.101}$$

Due to the symmetry of $\underline{U}$, $U_{12} = U_{21}$ and hence we can conclude from (1.101), that $\tan\omega = -1/2\kappa(t)$. It then follows that

$$[\underline{R}] = \frac{1}{\sqrt{4+K^2(t)}} \begin{bmatrix} 2 & \kappa(t) & 0 \\ -\kappa(t) & 2 & 0 \\ 0 & 0 & \sqrt{4+K^2(t)} \end{bmatrix}, \tag{1.102}$$

$$[\underline{U}] = \frac{1}{\sqrt{4+K^2(t)}} \begin{bmatrix} 2 & \kappa(t) & 0 \\ \kappa(t) & 2+\kappa(t)^2 & 0 \\ 0 & 0 & \sqrt{4+K^2(t)} \end{bmatrix}. \tag{1.103}$$

The principal values and principal directions of $\underline{U}$ are,

$$(\lambda_1, \lambda_2, \lambda_3) = (\lambda, \frac{1}{\lambda}, 1), \qquad \text{where } \lambda \equiv \frac{1}{2}(K + \sqrt{4+K^2}), \tag{1.104}$$

and

$$\underline{u}^1 = \frac{1}{\sqrt{1+\lambda^2}}\,(\underline{e}_1 + \lambda\,\underline{e}_2), \qquad \underline{u}^2 = \frac{1}{\sqrt{1+\lambda^2}}\,(-\lambda\,\underline{e}_1 \ + \ \underline{e}_2), \qquad \underline{u}^3 = \underline{e}_3. \tag{1.105}$$

It follows that an infinitesimal line element which parallel to the $\underline{u}^1$ in the reference configuration, will increase in length by λ while and infinitesimal line element which is parallel to $\underline{u}^2$ in the reference configuration, will decrease in length by $1/\lambda$.

1.11.8 Non-Homogeneous Deformations

Shearing Deformations

Recall, that in the section on homogeneous deformations, we considered isochoric plane strain (pure shear) and simple shear. For both these deformations, the principal invariants of $\underline{C}$ were of the form,

$$I = II \qquad \text{and} \qquad III = 1, \tag{1.106}$$

and for simple shear $I = 3 + \kappa^2$. We can generalize this to define shearing deformations (either homogeneous or non-homogeneous) as any deformations for which the principal invariants are of the from (1.106) where, in general, I may depend on the position in the body and time. As in simple shear, we define the *amount of shear*, κ, through $\kappa^2 = I - 3$. It follows that all isochoric plane strains, either homogeneous or non-homogeneous are shearing deformations.

Exercise 1.11.1 *Prove the relationships between the components of the vorticity tensor and vorticity vector given in (1.77).*

Exercise 1.11.2 *Use the transport theorem to provide an alternative proof that for an isochoric motion, the divergence of the velocity vector is zero.*

Chapter 2

Governing Equations

2.1 Governing Equations

Before discussing the governing equations, it is helpful to review the transport theorem.

The Transport Theorem

The transport theorem for an arbitrary scalar function ϕ of position $\underline{x}$ and time t is

$$\boxed{\frac{d}{dt}\int_{\mathcal{V}} \tilde{\phi}(\underline{x},t)\; dv \;=\; \int_{\mathcal{V}} \left(\frac{D\phi}{Dt} \;+\; \phi\,\mathrm{div}\underline{v} \right)\; dv} \tag{2.1}$$

where $\underline{v}$ is the velocity vector and $\mathcal{V}(t)$ be an arbitrary material volume of the body in the present configuration at time t.

Proof

Recall that the integral of a function $\phi(\underline{x}, t)$ over the an arbitrary material volume of the body in the present configuration at time t can be written with respect to an integral over the corresponding material volume $\mathcal{V}_0$ in the reference configuration κ_0

$$\int_{\mathcal{V}(t)} \tilde{\phi}(\underline{x},t)dv \;=\; \int_{\mathcal{V}_0} \bar{\phi}(\underline{X},t)JdV, \tag{2.2}$$

where $dv = JdV$. Taking the time derivative of (2.2), we find

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{V}} \tilde{\phi}(\underline{x},t)\; dv \quad &= \; \frac{d}{dt}\int_{\mathcal{V}_o} \left(\bar{\phi}(\underline{X},t)\; J\right) dV \\ &= \int_{\mathcal{V}_o} \left(\frac{D\phi}{Dt}\; J \;+\; \tilde{\phi}\,\frac{DJ}{Dt} \right)\; dV \\ &= \int_{\mathcal{V}_o} \left(\frac{D\phi}{Dt} \;+\; \phi\mathrm{div}\underline{v} \right) JdV \end{aligned} \tag{2.3}$$

where we have made use of (1.31) and the fact that $\mathcal{V}_o$ is independent of time. Eq. (2.1) follows directly from this result. Alternatively, it is sometimes convenient to write (2.3) as,

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \tilde{\phi}(\underline{x},t)dv \;=\; \int_{\mathcal{V}(t)} \frac{\partial\tilde{\phi}(\underline{x},t)}{\partial t}\,dv \;+\; \int_{\delta\mathcal{V}(t)} \tilde{\phi}(\underline{x},t)\underline{\tilde{v}}(\underline{x},t)\cdot\underline{n}dv \qquad (2.4)$$

where $\mathcal{V}(t)$ is bounded by a closed regular surface $\delta\mathcal{V}$ with outward normal $\underline{n}(\underline{x},t)$. In what follows, we will also make use of the following theorem:

Theorem:

If $\tilde{\phi}(\underline{x},t)$ is continuous in the region $\mathcal{R}(t)$, then a necessary and sufficient condition for

$$\int_{\mathcal{V}(t)} \tilde{\phi}(\underline{x},t)dv \;=\; 0 \quad \text{for every} \quad \mathcal{V}(t)\subseteq\mathcal{R}(t) \qquad (2.5)$$

is

$$\tilde{\phi}(\underline{x},t) \;=\; 0 \;\; \text{all } \underline{x} \text{ in } \mathcal{R}.$$

2.1.1 Conservation of Mass

The mass $\mathcal{M}$ of a section $\mathcal{V}(t)$ of the body at time t is

$$\mathcal{M} \;=\; \int_{\mathcal{V}(t)} \tilde{\rho}(\underline{x},t)\,dv, \qquad (2.6)$$

where ρ is the mass density of the body at time t. The mass $\mathcal{M}_0$ of the same material points in configuration κ_0 is,

$$\mathcal{M}_0 \;=\; \int_{\mathcal{V}_0} \rho_0(\underline{X})\,dV, \qquad (2.7)$$

where $\rho_0(\underline{X})$ is the mass density of the material in κ_0. The principle of conservation of mass is the postulate that the mass of the body in an arbitrary material volume $\mathcal{V}(t)$, does not change in time

$$\mathcal{M}(t) = \mathcal{M}_0 \qquad \text{for all time } t. \qquad (2.8)$$

Transforming the volume integral in (2.6) over $\mathcal{V}(t)$ to an integral over $\mathcal{V}_0$, using this result with (2.7) and (2.8), and making necessary continuity assumptions, we have directly that

$$\rho\, J \;=\; \rho_0 \qquad \text{all } \underline{X}\subseteq\mathcal{R}_0. \qquad (2.9)$$

This is the local Lagrangian form of conservation of mass. Alternatively, taking the time derivative of (2.8) and using (2.6) and (2.6), it follows that

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \tilde{\rho}(\underline{x},t)\,dv \;=\; 0. \qquad (2.10)$$

Making use of the Transport Theorem, we can write the principle of conservation of mass (2.10) as,

$$0 \;=\; \int_{\mathcal{V}(t)} \left(\frac{D\rho}{Dt} \;+\; \rho(\underline{x},t)\mathrm{div}\,\underline{v}(\underline{x},t) \right) dv. \tag{2.11}$$

Making suitable assumptions about continuity of the field variables and using (**??**), we obtain the local form of (2.11),

$$\frac{D\rho}{Dt} \;+\; \rho\,\mathrm{div}\,\underline{v} \;=\; 0 \qquad \text{all}\,\underline{x} \subseteq \mathcal{R} \tag{2.12}$$

or in indicial notation,

$$\frac{D\rho}{Dt} \;+\; \rho\,\frac{\partial \tilde{v}_i}{\partial x_i} \;=\; 0 \qquad \text{all }\underline{x} \subseteq \mathcal{R}. \tag{2.13}$$

Implications of Conservation of Mass for Incompressible Materials

The motion of an incompressible materials is constrained to be isochoric, in which case for **all** motions: the divergence of the velocity vector is *constrained* to be zero and J is one. In this case, the local form of the conservation of mass in Eulerian form, (2.31), reduces to,

$$\frac{D\rho}{Dt} \;=\; 0 \qquad \text{all }\underline{x} \subseteq \mathcal{R}. \tag{2.14}$$

and (2.9) becomes

$$\rho \;=\; \rho_0 \qquad\qquad \text{all }\underline{X} \subseteq \mathcal{R}_0. \tag{2.15}$$

Note that the even though the material derivative of the density it is zero, it is not necessary that the density be constant in space. For example, a stratified fluid with density distribution, $\rho = \rho_0 \,+\, \alpha y$, where α is a constant, can experience simple shear. Simple shear refers to the velocity field, $\underline{v} \;=\; \gamma\, y\underline{e}_x$. This flow field has particular importance because it the solution for steady, fully developed flow between two parallel plates driven by the motion of the upper plate $\underline{v} \;=\; U\underline{e}_x$. The lower plate is located in the plane $y = 0$ and the upper plate in the plane $y = h$. If the plates are separated by a distance h then $\gamma = U/h$. It is easily seen that this motion is isochoric, a necessary condition for an incompressible fluid to experience this motion. In addition,

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} \;+\; \frac{\partial\rho}{\partial x_i}\,v_i \;=\; 0 \tag{2.16}$$

for the stratified fluid and given motion and therefore conservation of mass is satisfied. As will be discussed at a later point, this motion is compatable with the balance of linear momentum for both inviscid and linearly viscous fluids. In fact, a much wider class of fluids can be shown to undergo this motion.

2.1.2 Balance of Linear Momentum

The postulate of balance of linear momentum is the statement that the rate of change of linear momentum of a fixed mass of the body is equal to the sum of the forces acting on the body. These forces can show up as body forces, or as forces due to stress vectors acting on the surface of the body,

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \rho \underline{v}\, dv \;=\; \int_{\mathcal{V}(t)} \rho\, \underline{b}\, dv \;+\; \int_{\delta\mathcal{V}(t)} \underline{t}\, da \tag{2.17}$$

where $\underline{b}$ is the body force per unit mass, $\underline{t}(\underline{x}, t, \underline{n})$ is the surface force acting on the body in the current configuration per unit area of $\delta\mathcal{V}$ and $\underline{n}$ is the unit normal to the surface $\delta\mathcal{V}$ at $\underline{x}$ at time t. Note that stress vector depends on position, time and the unit normal to the surface at $\underline{x}$. The first and second integrals on the right hand side of (2.17) represent the contributions due to body forces and to surface forces respectively.

It is somewhat problematic that the stress vector depends on the surface under consideration. Fortunately, we can show that it is possible (under certain conditions) to show that this dependence is linear. Fortunately, as discussed below, the existence of a second order tensor, $\underline{T}$ can be shown, where,

$$\underline{t} \;=\; \underline{\tilde{t}}(\underline{x}, t, \underline{n}) \;=\; \underline{\tilde{T}}(\underline{x}, t)\underline{n}. \tag{2.18}$$

and $\underline{T}$ is called the Cauchy stress tensor.

Cauchy's Lemma

Consider an arbitarry part of the material region of the body $\mathcal{B}$ which occupies a part $\mathcal{V}$ in the present configuration at time t with bounding surface $\partial\mathcal{V}$. Let $\mathcal{V}$ be divided into two regions $\mathcal{V}_1, \mathcal{V}_2$ separated by a surface σ. Further, let $\partial\mathcal{V}_1, \partial\mathcal{V}_2$ refer to the boundaries of $\mathcal{V}_1, \mathcal{V}_2$, respectively and let $\partial\mathcal{V}', \partial\mathcal{V}''$ be the portions of the boundaries of $\mathcal{V}_1, \mathcal{V}_2$ such that

$$\partial\mathcal{V}' \;=\; \partial\mathcal{V}_1 \cap \partial\mathcal{V}, \qquad \partial\mathcal{V}'' \;=\; \partial\mathcal{V}_2 \cap \partial\mathcal{V}. \tag{2.19}$$

Thus,

$$\mathcal{V} \;=\; \mathcal{V}_1 \cup \mathcal{V}_2, \qquad \partial\mathcal{V} \;=\; \partial\mathcal{V}' \cup \partial\mathcal{V}'' \qquad \partial\mathcal{V}_1 \;=\; \partial\mathcal{V}' \cup \sigma, \qquad \partial\mathcal{V}_2 \;=\; \partial\mathcal{V}'' \cup \sigma. \tag{2.20}$$

Now recall the balance of linear momentum for the arbitrary material region $\mathcal{V}$,

$$\frac{d}{dt}\int_{\mathcal{P}} \rho \underline{v}\, dv \;=\; \int_{\mathcal{V}} \rho\, \underline{b}\, dv \;+\; \int_{\partial\mathcal{V}} \underline{t}\, da. \tag{2.21}$$

The balance of linear momentum can also be considered separately for parts $\mathcal{V}_1$ and $\mathcal{V}_2$,

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{V}_1} \rho \underline{v}\, dv \;&=\; \int_{\mathcal{V}_1} \rho\, \underline{b}\, dv \;+\; \int_{\partial\mathcal{V}_1} \underline{t}\, da, \\ \frac{d}{dt}\int_{\mathcal{V}_2} \rho \underline{v}\, dv \;&=\; \int_{\mathcal{V}_2} \rho\, \underline{b}\, dv \;+\; \int_{\partial\mathcal{V}_2} \underline{t}\, da. \end{aligned} \tag{2.22}$$

as well as the union of these regions $\mathcal{V}_1 \cup \mathcal{V}_2$, be

$$\frac{d}{dt}\int_{\mathcal{V}_1\cup\mathcal{V}_2} \rho \underline{v}\, dv \;=\; \int_{\mathcal{V}_1\cup\mathcal{V}_2} \rho\, \underline{b}\, dv \;+\; \int_{\partial\mathcal{V}_1\cup\partial\mathcal{V}_2} \underline{t}\, da. \tag{2.23}$$

The stress vector $\underline{t}(\underline{x}, t; \underline{n})$ is $(2.22)_1$ acting over the boundary $\partial\mathcal{V}_1$ results from contact forces exerted by the material on one side of the boundary (exterior to $\partial\mathcal{V}_1$) on the material on the other side. Similar results hold for the stress vectors in $(2.22)_2$ and (2.23). It should be emphasized that the stress vector in $(2.22)_1$ over $\partial\mathcal{V}' \cup \sigma$ represents the contact force exerted in $\mathcal{V}_1$ across the surface. The approriate normals associated with $\underline{t}(\underline{x}, t; \underline{n})$ on σ are equal in magnitude and opposite in sign. In particular, if $\underline{n}$ is the outward normal at $\underline{x}$ on surface σ of $\mathcal{V}_1$, then the outward normal at $\underline{x}$ on surface σ of $\mathcal{V}_2$ is $-\underline{n}$.

After combining $(2.22)_1$ and $(2.22)_2$ and subtracting (2.23), we obtain,

$$\int_\sigma \left[\underline{t}(\underline{x}, t; \underline{n}) \;-\; \underline{t}(\underline{x}, t; -\underline{n})\right] da \;=\; 0. \tag{2.24}$$

Assuming that the stress vector is a continuous function of position and $\underline{n}$, it follows that

Cauchy's Lemma

$$\underline{t}(\underline{x}, t; \underline{n}) \;=\; -\,\underline{t}(\underline{x}, t; -\underline{n}). \tag{2.25}$$

The stress vectors acting on opposite sides of the same surface at a given point and time are equal in magnitude and opposite in sign.

Existence of the stress tensor and its relationship to the stress vector

At this point, it has not been shown in what way the stress vector depend on the normal to the surface. It can be shown that there exists a second order tensor $\underline{T}(\underline{x}, t)$ such that

$$\underline{t}(\underline{x}, t; \underline{n}) \;=\; \underline{T}(\underline{x}, t)\cdot\underline{n} \qquad \text{or} \qquad t_k(\underline{x}, t; \underline{n}) \;=\; T_{ki}(\underline{x}, t)\, n_i \tag{2.26}$$

The stress vector depends linearly on the normal to the surface.

The second order $\underline{T}$ is called the Cauchy stress tensor. Significantly, $\underline{T}$ is independent of the surface $\underline{n}$. Sometimes, it is convenient to write this result in matrix form,

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}. \tag{2.27}$$

So, for example, if we consider a surface with normal $\underline{n} = \underline{e}_1$, then

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and therefore} \qquad \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{21} \\ T_{31} \end{bmatrix}, \tag{2.28}$$

or, equivalently,

$$\underline{t} \;=\; T_{11}\underline{e}_1 \;+\; T_{21}\underline{e}_2 \;+\; T_{31}\underline{e}_3. \tag{2.29}$$

Making use of the divergence theorem as well as the relationship between the stress tensor and the stress vector, (2.17) can be written as,

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \rho\underline{v}\,dv \;=\; \int_{\mathcal{V}(t)} (\rho\underline{b} \;+\; \mathrm{div}\,\underline{T})\;dv. \tag{2.30}$$

The local form of the equation of linear momentum can be obtained from (2.30) by using the Transport Theorem and making suitable assumptions about the continuity of the field variables,

$$\rho\underline{a} \;=\; \rho\underline{b} \;+\; \mathrm{div}\,\underline{T} \qquad \text{all } \underline{x} \subseteq \mathcal{R} \tag{2.31}$$

or, using index notation,

$$\rho a_i \;=\; \rho b_i \;+\; \frac{\partial T_{ij}}{\partial x_j} \qquad \text{all } \underline{x} \subseteq \mathcal{R}, \tag{2.32}$$

where the acceleration is given in (1.13) and (1.18). It is often convenient to represent $\underline{T}$ as the sum of a deviatoric and spherical part.

$$\underline{T} \;=\; \underline{\tau} \;+\; \bar{t}\;\underline{I}, \tag{2.33}$$

where,

$$\tau_{ii} \;=\; 0, \text{ and } \bar{t} \;=\; \frac{1}{3}T_{kk}. \tag{2.34}$$

The tensor $\underline{\tau}$ is often referred to as the deviatoric part of $\underline{T}$ and $\bar{t}\underline{I}$ as the spherical part. When the Cauchy stress tensor is decomposed in this way, $-\bar{t}$ is often called the *pressure* and denoted by p. Using the decomposition, (2.33), the balance of linear momentum can be written as,

$$\rho\underline{a} \;=\; -\nabla p \;+\; \mathrm{div}\,\underline{\tau} \;+\; \rho\underline{b} \qquad \text{all } \underline{x} \subseteq \mathcal{R}. \tag{2.35}$$

Alternatively,

$$\rho a_i \;=\; -\frac{\partial p}{\partial x_i} \;+\; \frac{\partial \tau_{ij}}{\partial x_j} \;+\; \rho b_i \qquad \text{all } \underline{x} \subseteq \mathcal{R}. \tag{2.36}$$

For compressible fluids, p is the thermodynamic pressure and an equation of state is required, for example the ideal gas law. For incompressible fluids, p is a mechanical pressure arising from the constraint of incompressibility. No equation of state is necessary, rather, p will be determined as part of the solution to the governing equations and boundary conditions.

Piola-Kirchhoff Stress Tensor

We now consider $\underline{p}$, the surface force acting on the body in the current configuration per unit area of $\delta\mathcal{V}_0$. The vectors $\underline{p}$ and $\underline{t}$ are related through,

$$\int_{\delta\mathcal{V}_0} \underline{p}\,dA \;=\; \int_{\delta\mathcal{V}} \underline{t}\,da. \tag{2.37}$$

The balance of linear momentum can also be written with respect to $\delta\mathcal{V}_0$ and $\mathcal{V}_0$ through the use of $dv = JdV$ and $\rho J = \rho_0$ as

$$\int_{\mathcal{V}_0} \rho_0 \underline{a}\, dV \;=\; \int_{\mathcal{V}_0} \rho_0\, \underline{b}\, dV_0 \;+\; \int_{\delta\mathcal{V}_0} \underline{p}\, dA. \tag{2.38}$$

Similar to the discussion for $\underline{t}$, it can be shown that

$$\underline{\bar{p}}(\underline{X}, t, \underline{N}) \;=\; -\underline{\bar{p}}(\underline{X}, t, \underline{N}) \tag{2.39}$$

and

$$\underline{\bar{p}}(\underline{X}, t, \underline{N}) \;=\; \underline{\bar{P}}(\underline{X}, t)\underline{N}, \tag{2.40}$$

where $\underline{P}$ is the first Piola-Kirchoff stress tensor and $\underline{N}$ is the unit normal to the surface $\delta\mathcal{V}$ at $\underline{X}$. It then follows from (2.40) and the relationship between $\underline{da} \;=\; J\underline{F}^{-T}\, \underline{dA}$, that,

$$\underline{P} \;=\; J\, \underline{T}\, \underline{F}^{-T} \qquad \text{all } \underline{X} \subseteq \mathcal{R}_0. \tag{2.41}$$

The first Piolo-Kirchoff tensor is not symmetric. It is often convenient for numerical purposes to define a symmetric stress tensor $\underline{S}$, called the second Piola-Kirchoff stress, related to $\underline{P}$ and $\underline{T}$ through

$$\underline{S} \;=\; \underline{F}^{-1}\underline{P}, \qquad \text{and} \qquad \underline{S} \;=\; J\, \underline{F}^{-1}\underline{T}\underline{F}^{-T}. \tag{2.42}$$

Making use of (2.40) in , it follows from (2.38) that,

$$\int_{\mathcal{V}_0} \rho_0 \underline{a}\, dV \;=\; \int_{\mathcal{V}_0} (\rho_0\, \underline{b} \;+\; \text{Div}\underline{P})\; dV. \tag{2.43}$$

The corresponding local form of (2.43) is,

$$\rho_0 \underline{a} \;=\; \rho_0\, \underline{b} \;+\; \text{Div}\underline{P} \qquad \text{all } \underline{X} \subseteq \mathcal{R}_0. \tag{2.44}$$

2.1.3 Balance of Angular Momentum

The balance of angular momentum is the statement that the rate of change of moment of momentum of a material volume is equal to the sum of the all the moments acting on the part of the body. The integral form of the balance of angular momentum, is in the absence of body couples,

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \rho \underline{x} \times \underline{v}\, dv \;=\; \int_{\mathcal{V}(t)} \rho\, \underline{x} \times \underline{b}\, dv \;+\; \int_{d\mathcal{V}(t)} \underline{x} \times \underline{t}\, da. \tag{2.45}$$

In writing (2.45), we have assumed there are no body couples $\underline{c}$ per unit volume acting on the body which would contribute an additional torque,

$$\int_{\mathcal{V}(t)} \rho\, \underline{c}\, dv. \tag{2.46}$$

Making use of the balance of linear momentum, it can be shown that (2.45) reduces to the requirement that the Cauchy stress tensor must be symmetric,

$$\underline{T} \;=\; \underline{T}^T \qquad \text{all } \underline{x} \subseteq \mathcal{R}. \tag{2.47}$$

The balance of angular momentum can also be written with respect to $\delta\mathcal{V}_0$ and $\mathcal{V}_0$ through the use of $dv = JdV$ and $\rho J = \rho_0$ as

$$\int_{\mathcal{P}_0} \rho_0 \underline{x} \times \underline{a}\, dV \;=\; \int_{\mathcal{P}_0} \rho_0\, \underline{x} \times \underline{b}\, dV \;+\; \int_{\delta\mathcal{P}} \underline{x} \times \underline{p}\, dA, \tag{2.48}$$

The corresponding local form is,

$$\underline{P}\underline{F}^T \;=\; \underline{F}\underline{P}^T \qquad \text{all } \underline{X} \subseteq \mathcal{R}_0 \tag{2.49}$$

Notice that $\underline{p}$ is not in general symmetric. The second Piola-Kirchhoff stress tensor, $\underline{S}$ is defined through,

$$\underline{S} \;=\; \underline{F}^{-1}\underline{P} \qquad \text{or} \qquad S_{AB} = F^{-1}_{Ai}\, P_{iB} \tag{2.50}$$

all $\underline{X}_0 \subseteq \mathcal{R}_0$ and hence it follows from (2.48) that $\underline{S}$ is symmetric. Additionally, this means,

$$\underline{S} \;=\; J\underline{F}^{-1}\underline{T}\underline{F}^{-T} \qquad \text{or} \qquad S_{AB} \;=\; JF^{-1}_{iA} T_{ij} F^{-1}_{Bj}. \tag{2.51}$$

2.1.4 Mechanical Energy Equation

It is sometimes useful to consider the *Mechanical Energy Equation* which is obtained from the equation of linear momentum making use of the equation of conservation of mass. If we take the inner product of the velocity vector and the local form of the equation of linear momentum (2.32) we obtain,

$$\frac{1}{2}\rho \frac{D}{Dt}(v_i v_i) \;=\; \frac{\partial T_{ij}}{\partial x_j} v_i \;+\; \rho b_i v_i \qquad \text{all } \underline{x} \subseteq \mathcal{R}. \tag{2.52}$$

An integral form of this equation can be obtained by integrating (2.52) over a fixed region of the body (same material particles) occupying region $\mathcal{V}(t)$ with surface $\partial\mathcal{V}(t)$, to obtain,

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \frac{1}{2}\rho \underline{v} \cdot \underline{v} dv \;=\; \int_{\delta\mathcal{V}(t)} \underline{t} \cdot \underline{v} da \;-\; \int_{\mathcal{V}(t)} \underline{T} : \underline{D} dv \;+\; \int_{\mathcal{V}(t)} \rho \underline{b} \cdot \underline{v} dv. \tag{2.53}$$

where we have made use of the divergence theorem and the conservation of mass. For future referece, we introduce the following notation for terms in (2.55):

$$\begin{aligned}
\mathcal{K} &\equiv \int_{\mathcal{V}(t)} \frac{1}{2}\rho\underline{v}\cdot\underline{v}dv &&= \text{kinetic energy in } \mathcal{V}(t)\\
R_c &\equiv \int_{\delta\mathcal{V}(t)} \underline{t}\cdot\underline{v}da &&= \text{rate of work done by surface forces on the boundary } \delta\mathcal{V}(t)\\
R_b &\equiv \int_{\mathcal{V}(t)} \rho\underline{b}\cdot\underline{v}dv &&= \text{rate of work done on the material volume } \mathcal{V}(t) \text{ by body forces}\\
\mathcal{P}_{int} &\equiv \int_{\mathcal{V}(t)} \underline{T}:\underline{D}dv &&= \text{rate of internal work in the material volume } \mathcal{V}(t).
\end{aligned} \tag{2.54}$$

The scalar, $\underline{T}:\underline{D}$ is the rate of work by stresses per unit volume of the body and is often called the *stress power*. The sum of R_c and R_b is then the rate of work by external forces on the material volume.

In summary, we can write the mechanical energy equation as

Mechanical Energy Equation

$$\frac{d}{dt}\mathcal{K} + \mathcal{P}_{int} = R_c + R_b. \tag{2.55}$$

Alternatively, we can start with the local form of linear momentum in Lagrangian form, take the inner product with the velocity and integrate over the corresponding material region $\mathcal{V}_0$ in κ_0. After using the divergence theorem, we can show that we obtain the same form (2.55) with

$$\mathcal{K} = \int_{\mathcal{V}_0} \frac{1}{2}\rho_0\underline{v}\cdot\underline{v}dV, \quad R_c = \int_{\delta\mathcal{V}_0} \underline{p}\cdot\underline{v}dA, \quad R_b = \int_{\mathcal{V}_0} \rho_0\underline{b}\cdot\underline{v}dV, \quad \mathcal{P}_{int} = \int_{\mathcal{V}_0} \underline{P}:\dot{\underline{F}}dV. \tag{2.56}$$

In fact, this last quantity can also be written with respect to $\underline{T}$, $\underline{P}$, or $\underline{S}$,

$$\mathcal{P}_{int} = \int_{\mathcal{V}(t)} \underline{T}:\underline{D}dv = \int_{\mathcal{V}_0} \underline{P}:\dot{\underline{F}}dV = \int_{\mathcal{V}_0} \underline{S}:\dot{\underline{E}}dV. \tag{2.57}$$

2.1.5 Balance of Energy

If we consider a part of the body $\mathcal{V}(t)$ in the current configuration, we can hypothesize the existence of a scalar called the specific internal energy, $u = u(\underline{x}, t)$ (internal energy per unit mass). The internal energy for the part $\mathcal{V}(t)$ of the body, will then be

$$\int_{\mathcal{V}(t)} \rho u dv. \tag{2.58}$$

We do not go delve into a discussion of entropy here, but remind the reader that the specific internal energy is related to the Helmholtz potential ψ and the entropy η through,

$$u = \psi + \eta T \tag{2.59}$$

where T is the absolute temperature. Recall that the kinetic energy of the part $\mathcal{V}(t)$of the body is

$$\int_{\mathcal{V}(t)} \frac{1}{2}\rho \underline{v} \cdot \underline{v} dv. \tag{2.60}$$

Heat may enter the body through the surface $d\mathcal{V}(t)$ of the body with outward unit normal $\underline{n}$. It can be shown, that this heat flux can be represented as the scalar product of a vector $\underline{q}$ and the normal to the surface $\underline{n}$. Where $\underline{q} \cdot \underline{n}$ positive is associated with heat leaving the surface and $\underline{q} \cdot \underline{n}$ negative is associated with heat entering the surface. In addition, heat may enter the body as a specific heat supply, $r = r(\underline{x}, t)$: the heat entering the body per unit mass per unit time. Therefore the rate of heat entering the part $\mathcal{V}(t)$ of the body is

$$-\int_{d\mathcal{V}(t)} \underline{q} \cdot \underline{n} da + \int_{\mathcal{V}(t)} \rho r dv. \tag{2.61}$$

The balance of energy is a statement that the rate of increase in internal energy and kinetic energy in the part $\mathcal{V}(t)$ of the body is equal to the rate of work by body forces and contact forces plus energies due to heat entering the body per unit time. We can write this statement as

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \rho(u + \frac{1}{2}\underline{v} \cdot \underline{v})dv = \int_{d\mathcal{V}(t)} \underline{t} \cdot \underline{v} da + \int_{\mathcal{V}(t)} \rho \underline{b} \cdot \underline{v} dv - \int_{d\mathcal{V}(t)} \underline{q} \cdot \underline{n} da + \int_{\mathcal{V}(t)} \rho r dv \tag{2.62}$$

where the first integral on the left hand side is the rate of work by contact forces, the second integral is the rate of work by body forces. Making suitable assumptions about continuity of the field variables, we can obtain the local form of (2.62),

$$\rho\left(\frac{Du}{Dt} + \underline{v} \cdot \frac{D\underline{v}}{Dt}\right) = \underline{T} : \underline{D} + \underline{v} \cdot (\text{div}\underline{T}) + \rho \underline{v} \cdot \underline{b} - \text{div}\underline{q} + \rho r. \tag{2.63}$$

Using results from the mechanical energy equation, we can rewrite (2.63) as

Local, Eulerian form of Balance of Energy

$$\rho\frac{Du}{Dt} = \underline{T} : \underline{D} - \text{div}\underline{q} + \rho r,$$

or

$$\rho\frac{Du}{Dt} = T_{ij}D_{ji} - \frac{\partial q_i}{\partial x_i} + \rho r. \tag{2.64}$$

Similarly, we can write the Lagrangian form of the balance of energy,

Local, Lagrangian form of Balance of Energy

$$\rho_0 \dot{u} \;=\; \underline{P}^T : \dot{\underline{F}} \;-\; \text{Div}\underline{q}_0 \;+\; \rho_0 r,$$

or

$$\rho_0 \dot{u} \;=\; P_{iA}\dot{F}_{iA} \;-\; \frac{\partial q_{0_A}}{\partial X_A} \;+\; \rho_0 r. \tag{2.65}$$

where

$$\int_{\mathcal{V}(t)} \underline{q} \cdot \underline{da} \;=\; \int_{\mathcal{V}_0} \underline{q}_0 \cdot \underline{dA}. \tag{2.66}$$

2.1.6 Clausius-Duhem Inequality

The second law of thermodynamics is formulated in a variety of ways. The Eulerian form of the Clausius-Duhem inequality is,

Eulerian form of the Clausius Duhem Inequality

Finite Volume Form

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \rho\eta dv \geq \int_{\mathcal{V}(t)} \frac{\rho r}{T} dv \;-\; \int_{\mathcal{V}(t)} \frac{\underline{q}\cdot\underline{n}}{T} da \tag{2.67}$$

The local form is,

$$\dot{\eta} \;-\; \frac{r}{T} \;+\; \frac{1}{\rho}\text{div}(\frac{\underline{q}}{T}) \geq 0. \tag{2.68}$$

For cases where we use the temperature as an independent variable, it is convenient to make use of $\psi = u - T\eta$, in which case, we can write this last result as,

$$-\rho\left(\dot{\psi} \;+\; \eta\dot{T}\right) \;+\; \underline{T} : \underline{D} \;-\; \frac{1}{T}\underline{q} \cdot \text{grad}\, T \geq 0. \tag{2.69}$$

The Lagrangian form of (2.67) is,

Lagrangian form of the Clausius Duhem Inequality

Finite Volume Form

$$\frac{d}{dt}\int_{\mathcal{V}_0} \rho_0\eta dV \geq \int_{\mathcal{V}_0} \frac{\rho_0 r}{T} dV \;-\; \int_{\mathcal{V}_0} \frac{\underline{q}\cdot\underline{N}}{T} dA \tag{2.70}$$

$$\dot{\eta} \;-\; \frac{r}{T} \;+\; \frac{1}{\rho_0}\text{div}(\frac{\underline{q}_0}{T}) \geq 0. \tag{2.71}$$

and the corresponding local form is,

$$-\rho_0\left(\dot{\psi} \;+\; \eta\dot{T}\right) \;+\; \underline{P}^T : \dot{\underline{F}} \;-\; \frac{1}{T}\underline{q}_0 \cdot \text{Grad}\, T \geq 0. \tag{2.72}$$

It follows directly from (2.69) and (2.72) that for isothermal processes with no heat transfer ($T =$ constant and $\underline{q} = \underline{q}_0 = 0$), the Clausius Duhem Inequality reduces to,

Isothermal form of the Clausius Duhem Inequality

Eulerian

$$-\rho\,\dot{\psi} \;+\; \underline{T} : \underline{D} \geq 0, \tag{2.73}$$

Lagrangian

$$-\rho_0\,\dot{\psi} \;+\; \underline{P}^T : \dot{\underline{F}} \geq 0.$$

2.1.7 Summary of Governing Equations:

[1] **Integral Form for Finite Region $\mathcal{V}(t)$ with surface $\delta\mathcal{V}$, Fixed Material Particles** where $\mathcal{V}(t)$ is an arbitrary material volume such that $\mathcal{V}(t) \subseteq \mathcal{R}(t)$.

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{V}(t)} \rho\, dv \;&=\; 0. \\
\frac{d}{dt}\int_{\mathcal{V}(t)} \rho\underline{v}\, dv \;&=\; \int_{\mathcal{V}(t)} \rho\,\underline{b}\, dv \;+\; \int_{\delta\mathcal{V}(t)} \underline{t}\, da \\
\frac{d}{dt}\int_{\mathcal{V}(t)} \rho\underline{x} \times \underline{v}\, dv \;&=\; \int_{\mathcal{V}(t)} \rho\,\underline{x} \times \underline{b}\, dv \;+\; \int_{\delta\mathcal{V}(t)} \underline{x} \times \underline{t}\, da.
\end{aligned} \tag{2.74}$$

Corresponding Local Form

$$\left.\begin{aligned}
&\frac{D\rho}{Dt} \;+\; \rho\mathrm{div}\,\underline{v} \;=\; 0 \\
&\rho\underline{a} \;=\; \rho\underline{b} \;+\; \mathrm{div}\underline{T} \\
&\underline{T} \;=\; \underline{T}^T \\
&\rho\frac{Du}{Dt} \;=\; \underline{T} : \underline{D} \;-\; \mathrm{div}\underline{q} \;+\; \rho r \\
&-\rho\left(\dot{\psi} \;+\; \eta\dot{T}\right) \;+\; \underline{T} : \underline{D} \;-\; \frac{1}{T}\underline{q}\cdot\mathrm{grad}\,T \geq 0.
\end{aligned}\right\} \quad \text{all } \underline{x} \subseteq \mathcal{R} \tag{2.75}$$

Integral Form for Finite Region $\mathcal{V}_0$ with surface $\delta\mathcal{V}_0$, Fixed Material Particles. where $\mathcal{V}_0$ is an arbitrary material volume such that $\mathcal{V}_0 \subseteq \mathcal{R}_0$. Note, these equations are for the current time, but the integrals are written for areas and volumes in the reference configuration. The density used in these equations, ρ_0, is the density of the body when it

[1] This section will be updated with energy equation and clausius duhem inequality

is in the reference configuration. The stress vector $\underline{p}$ is the surface force acting on the body in the current configuration per unit area of $\delta\mathcal{V}_0$.

$$\begin{aligned}
\int_{\mathcal{V}_0} \rho_0(\underline{X})\, dV \;&=\; \int_{\mathcal{V}(t)} \tilde{\rho}(\underline{x}, t)\, dv \\
\int_{\mathcal{V}} \rho_0 \underline{a}\, dV \;&=\; \int_{\mathcal{V}} \rho_0\, \underline{b}\, dV \;+\; \int_{\delta\mathcal{V}} \underline{p}\, dA \qquad (2.76) \\
\int_{\mathcal{V}} \rho_0 \underline{x} \times \underline{a}\, dV \;&=\; \int_{\mathcal{V}} \rho_0\, \underline{x} \times \underline{b}\, dV \;+\; \int_{\delta\mathcal{V}} \underline{x} \times \underline{p}\, dA
\end{aligned}$$

Corresponding Local Form

$$\left.\begin{aligned}
\rho\, J \;&=\; \rho_0 \\
\rho_0 \underline{a} \;&=\; \rho_0\, \underline{b} \;+\; \mathrm{Div}\underline{P} \\
\underline{P}\underline{F}^T \;&=\; \underline{F}\underline{P}^T
\end{aligned}\right\} \qquad \text{all } \underline{X} \subseteq \mathcal{R}_0 \qquad (2.77)$$

Integral Form for Fixed Finite Region CV with surface CS

$$\begin{aligned}
&\frac{d}{dt}\int_{CV} \rho\, dv \;+\; \frac{d}{dt}\int_{CS} \rho\, \underline{v} \cdot \underline{n}\, da \;=\; 0 \\
&\frac{d}{dt}\int_{CV} \rho\, \underline{v} dv \;+\; \int_{CV} \rho\, \underline{v}\, \underline{v} \cdot \underline{n} dv \;=\; \int_{CV} \rho\, \underline{b}\, dv \;+\; \int_{CS} \underline{t}\, da \qquad (2.78) \\
&\frac{d}{dt}\int_{CV} \rho\, \underline{x} \times \underline{v}\, dv \;+\; \int_{CV} \rho\, \underline{x} \times \underline{v}\, \underline{v} \cdot \underline{n} dv \;=\; \int_{CV} \rho\, \underline{x} \times \underline{b}\, dv \;+\; \int_{CS} \underline{x} \times \underline{t}\, da
\end{aligned}$$

[2]

2.2 Constraints

2.2.1 Incompressible materials - constraint of incompressibility

Suppose a material is constrained to only undergo isochoric motions, so that $J = 1$ at all points in the body for all times during any motion. We call such a material incompressible. If we take the material derivative of the equation $J = 1$, we find

$$0 \;=\; \frac{DJ}{Dt} \;=\; J\mathrm{div}\underline{v} \;=\; JD_{ii}. \qquad (2.79)$$

For such a material, we will assume the stress tensor has two contributions, a part $\underline{N}$ which does no work for motions of the form (2.79), and an extra stress $\underline{T}_E$ which requires a constitutive equation,

$$\underline{T} \;=\; \underline{T}_E \;+\; \underline{N}. \qquad (2.80)$$

[2]Insert section here about state of stress

Namely, we require

$$\underline{N} : \underline{D} \;=\; 0, \qquad \text{for all symmetric tensors } \underline{D} \text{ satisfying } \underline{D} : \underline{I} \;=\; 0. \tag{2.81}$$

Since $\underline{A} : \underline{B} = \text{tr}\underline{AB}$ defines an inner product in the six-dimensional space of symmetric tensors, we can restate (2.81) as a geometric requirement that $\underline{N}$ be orthogonal to all $\underline{D}$ which are orthogonal to $\underline{I}$. Necessarily, $\underline{N}$ is of the form,

$$\underline{N} \;=\; -p\underline{I} \tag{2.82}$$

where p is an undetermined Lagrange multiplier (scalar). Therefore, the Cauchy stress for an incompressible material will be defined through a constitutive equation of the form,

$$\underline{T} \;=\; \underline{T}_E - p\underline{I}. \tag{2.83}$$

2.2.2 Inextensible Fibers - constraint of inextensibility

Suppose we have a material in which infinitesimal material elements that are parallel to $\underline{a}_0$ in reference configuration κ_0 are constrained to be inextensible during the deformation. For a general motion, an infinitesimal material element $dX\underline{a}_0$ in κ_0 will be mapped to $dx\underline{a} = dX\underline{F}\,\underline{a}_0$ in $\kappa(t)$ where $\underline{a}_0$ and $\underline{a}$ are chosen to have unit length. For this constrained motion, $dx/dX = 1$, so that $\underline{a} = \underline{F}\underline{a}_0$. It therefore follows that $\underline{a}_0 \cdot \underline{C}\underline{a}_0 = 1$ for the entire deformation.

If we take the material derivative of this expression, it follows that, for such inextensible materials,

$$\begin{aligned} 0 &= \frac{D}{Dt}(\underline{a}_0 \cdot \underline{C}\underline{a}_0) \\ &= a_{0A}(L_{ij}F_{jA}F_{iB} + F_{iA}L_{ij}F_{jB})a_{0B} \\ &= 2\,D_{ij}\,F_{jA}\,a_{0A}\,F_{iB}\,a_{0B} \\ &= 2\,D_{ij}\,a_j a_i \\ &= 2\underline{D} : (\underline{a} \otimes \underline{a}) \end{aligned} \tag{2.84}$$

We assume, a material with inextensible fibers in the $\underline{a}_0$ direction in κ_0, has two contributions to the stress tensor, N, that does no work in motions satisfying the inextensibility constraint and an extra stress $\underline{T}_E$ which will be defined through a constitutive equation. In this case,

$$\underline{N} : \underline{D} \;=\; 0, \qquad \text{for all symmetric tensors } \underline{D} \text{ satisfying } \underline{D} : (\underline{a} \otimes \underline{a}) \;=\; 0. \tag{2.85}$$

Since $\underline{A} : \underline{B} = \text{tr}\underline{AB}$ defines an inner product in the six-dimensional space of symmetric tensors, we can restate (2.84) as a geometric requirement that $\underline{N}$ be orthogonal to all $\underline{D}$ which are orthogonal to $\underline{a} \otimes \underline{a}$. Necessarily, $\underline{N}$ can be written as,

$$\underline{N} \;=\; -q\underline{a} \otimes \underline{a} \tag{2.86}$$

where q is an undetermined Lagrange multiplier (scalar), so that

$$\underline{T} \;=\; \underline{T}_E - q\underline{a} \otimes \underline{a}. \tag{2.87}$$

Chapter 3

Constitutive Equations

3.1 Restrictions on the Stress Tensor

Thus far, we have discussed governing equations fundamental to any continuum material. To close this system of governing equations, we need to select constitutive models for the material of interest. We choose to take a classical approach to this subject, whereby we start with general forms of the constitutive equation and then use fundamental principles to reduce the class of acceptable constitutive models. Prior to turning attention to the constitutive theories, we briefly summarize requirements imposed on constitutive models in order that they be deemed "physically reasonable". Here, and in the remainder of this chapter, we focus attention on purely mechanical theories, where, for example, the effect of temperature variations are negligible. We will disregard any non-mechanical influences and assume the state of the body is determined solely by the kinematical history, (e.g. page 56-68 [30]). Motivated by applications to blood flow, in later sections, attention will be concentrated on incompressible materials.

Principle of coordinate invariance

The constitutive equations must be independent of the coordinate system used to describe the motion of the body.

Principle of determinism for the stress

The stress in a body at the current time is determined by the history of motion of that body and independent of any aspect of its future behavior, [19].

Principle of local action

The determination of the stress for a given particle in the body is independent of the motion outside an arbitrary neighborhood of that particle (see [30], page 57 for a mathematical description of this principle). This principle was originally combined with the previous principle [19].

Principle of equipresence

Under this principle, a quantity which appears as an independent variable in one constitutive equation should be present in all others for that material unless it violates some law of physics or rule of invariance (e.g. page 359-360 [30] for an example in the context of thermoelasticity and historical discussion of this principle).

Principle of material frame indifference

There are two separate principles which embody the concept that the response of the material should be unaffected by its location and orientation. In the first, the mechanical response of a body is required to be unchanged under a superposed rigid body motion of the body if the change in orientation and position of the body is accounted for (e.g. [11] and pages 484-486 of [18]). The second principle is the requirement that the material response should be invariant under an arbitrary change of observer. For historical reasons, Truesdell and Noll refer to the first of these principles as the *Hooke-Poisson-Cauchy form* and the second as the *Zaremba-Jaumann form*. Strictly speaking, these two principles are different, the second being more restrictive since it includes improper orthogonal transformations, such as reflections. Truesdell and Noll provide an interesting discussion of the history of these two principles in [30], pages 45-47.

As will be discussed in the remainder of this chapter, invariance requirements play an important role in continuum mechanics in restricting the form of constitutive equations.

Thermodynamic restrictions

The second law of thermodynamics is the restriction that the total entropy production for all thermodynamic processes is never negative. In the remainder of this chapter, we restrict attention to purely mechanical theories for which thermal effects are negligible. In this case, this restriction can be reduced to the statement that the stress power be non-negative [17, 28, 29],

$$\underline{T} : \underline{D} \geq 0. \tag{3.1}$$

For example, in the case of an incompressible, Newtonian fluid $\underline{T} = -p\underline{I} + 2\mu\underline{D}$ and this condition can be shown to reduce to the requirement that $\mu \geq 0$.

Well-posedness

The system of governing equations for the purely mechanical theory arising from the conservation of mass, balance of linear momentum and the constitutive equation for the stress tensor should be well-posed. By this we mean existence, uniqueness and continuous dependence of the solution on the data can be shown (see, e.g. [12]).

Stability of the rest state

One of the methods used to evaluate the range of physically reasonable parameters for a material is to evaluate the conditions under which the rest state is stable. It seems physically reasonable to exclude ranges of material parameters for which the rest state is unstable to infinitesimal disturbances. This criterion has been used, for example, for viscoelastic fluids (see, e.g. [4, 15, 8]) as well as fiber fluid mixtures [9].

Attainability

An additional test, which is relatively straightforward and does not require formulation within the context of thermodynamics, is to evaluate the *attainability* of solutions for chosen benchmark flows for fluids or equilibrium deformations for solids. A constitutive equation would seem to be physically unreasonable if a chosen steady or time-periodic motion (e.g. steady Couette flow) is unattainable, no matter how gradually the driving mechanism is ramped to a constant value and no matter how small this constant value is. Attainability of "physically reasonable" steady flows has been studied for Newtonian fluids (e.g. [6, 13, 7] and the literature there cited) and well as for some viscoelastic fluids [22]. We emphasize that attainability of a given solution should not be confused with an examination of the stability of this solution (e.g. [6]).

Mechanical response of real materials

Experiments on real materials also provide restrictions on the range of material parameters that are physically reasonable. Since we cannot test every material in existence, strictly speaking, we cannot actually "prove" an experimentally based restriction on a constitutive equation is necessary. Rather, experimental results for certain categories of real materials (e.g. polymeric fluids) provide guidelines for defining a "reasonable" range of parameters for a given material.

By way of example, in this subsection, we turn attention to some restrictions we might impose on a class of fluids called incompressible simple fluids. Briefly, an incompressible simple fluid is a material for which the stress at a point and the current time is determined up to a pressure once the strain of each past configuration relative to the present configuration is known (e.g. [3]). Namely, unlike solid materials, we do not need to know the strain relative to some inherent "natural" configuration. As discussed earlier in this chapter, the density of incompressible materials is unchanged during the deformation.

It can be shown that the mechanical behavior of a chosen (but arbitrary) incompressible simple fluid is completely determined in some flows, for example simple shear flow, once three material functions are known for that fluid. This result is somewhat unexpected since an incompressible simple fluid may have many more than three material functions or constants. By material functions, we mean functions that depend only on the nature of the material, not, for example on the experimental conditions. For reasons to be described below, we will refer to these three functions as *viscometric functions.*

The three viscometric functions can be defined relative to the rectangular components of the Cauchy stress tensor T_{ij} given in (1.97) for simple shear,

$$\tau(\dot{\gamma}_0) \;=\; T_{12} \qquad \mathcal{N}_1(\dot{\gamma}_0) \;=\; T_{11} - T_{22} \qquad \mathcal{N}_2(\dot{\gamma}_0) \;=\; T_{22} - T_{33}. \tag{3.2}$$

We will refer to the functions $\tau(\dot{\gamma}_0), \mathcal{N}_1(\dot{\gamma}_0), \mathcal{N}_2(\dot{\gamma}_0)$ as the *shear stress function*, *first normal stress difference* and the *second normal stress difference*, respectively [1]. It can be shown that $\tau(\dot{\gamma}_0)$ is an odd function while $\mathcal{N}_1(\dot{\gamma}_0)$ and $\mathcal{N}_2(\dot{\gamma}_0)$ are even functions (e.g. pages 70-71 of [23]).

Alternatively, we can consider the viscometric functions η, ψ_1, ψ_2

$$\eta(\dot{\gamma}_0) \;=\; \frac{T_{12}}{\dot{\gamma}_0} \qquad \psi_1(\dot{\gamma}_0) \;=\; \frac{T_{11} - T_{22}}{\dot{\gamma}_0^2} \qquad \psi_2(\dot{\gamma}_0) \;=\; \frac{T_{22} - T_{33}}{\dot{\gamma}_0^2}, \tag{3.3}$$

referred to as the *viscosity*, *first normal stress coefficient* and *second normal stress coefficient*, where $\dot{\gamma}_0 \neq 0$. For Newtonian fluids, ψ_1 and ψ_2 are identically zero.

It turns out that simple shear is not the only motion for which the behavior of an incompressible simple fluid is completely determined once η, ψ_1 and ψ_2 are known. While most flows do not meet this requirement, there are a number of other flows, called *viscometric flows*, which do (see, e.g. [3]). Viscometric flows include steady, fully developed flow in a straight pipe of constant circular cross section (sometimes called Poiseuille flow) and steady, unidirectional flow between two concentric circular cylinders driven by the rotation of one or both of the cylinders about their common axis (sometimes called Couette flow). Most rheometers are designed to generate viscometric flows and can be used to measure one or more of these material functions.

Based on experimental data for real polymeric fluids, the sign of ψ_1 is expected to be non-negative and the sign of ψ_2 to be non-positive [2]. In addition, the ratio of the magnitude of the second normal stress coefficient to the first normal stress coefficient is commonly believed to be less than one half (e.g. [16, 21]).

Further Comments

Roughly speaking, the five principles given in this subsection restrict the general functional form of the constitutive equation (e.g. [**?**] for further details and historical information). The latter requirements impose restrictions on the range of parameters for specific constitutive models. In the following sections, applications of these restrictions to particular constitutive equations are considered.

3.1.1 Superposed Rigid Body Motion

Because of its importance in restricting the form of constitutive equations, we now discuss in detail the requirement that constitutive equations be invariant to a superposed rigid body motion. The fundamental idea is that the constitutive response should be invariant to a rigid motion superposed on the original motion.

[1] There is some variation in the literature for the definitions of normal stress and even the sign of the stress tensor. See page 71 of [23] for a nice discussion of this issue.

Kinematics for a superposed rigid body motion

In this section, we consider two motions of a body: an arbitrary motion

$$\underline{x} \;=\; \underline{\chi}(\underline{X}, t) \tag{3.4}$$

and a second motion which differs from the first by a superposed rigid body motion and possibly a time shift,

$$\underline{x}^+ \;=\; \underline{\chi}^+(\underline{X}, t^+). \tag{3.5}$$

Namely, the position of a material point which at time t is at position $\underline{x}$ in the first motion, is at position $\underline{x}^+$ in the second motion at time $t^+ = t + \alpha$. Here α is a constant. These are two separate motions as seen by the same observer. The spatial description of (3.5) is then,

$$\underline{x}^+ \;=\; \hat{\underline{\chi}}^+(\underline{x}, t). \tag{3.6}$$

In a rigid motion the distances between material points in the body are kept constant. Since the second motion differs from the first motion by a rigid body motion, the function $\hat{\underline{\chi}}^+(\underline{x}, t)$ must belong to a restricted class of motions.

It can be shown (see appendix), that the most general form for (3.6) is,

<u>Most General Superposed Rigid Body Motion</u>

$$\underline{x}^+ \;=\; \underline{c}(t) \;+\; \underline{Q}(t) \cdot \underline{x}, \tag{3.7}$$

where $\underline{Q}(t)$ is a proper orthogonal second order tensor,

or

$$x_i^+ \;=\; c(t)_i \;+\; Q(t)_{ij}\, x_j. \tag{3.8}$$

where $\underline{Q}$ has the properties,

$$\underline{Q} \cdot \underline{Q}^T \;=\; \underline{I}, \qquad\qquad \det \underline{Q} \;=\; +1. \tag{3.9}$$

The vector $\underline{c}$ can be interpreted as a rigid body translation and $\underline{Q}$ as a rotation tensor.

Transformation of kinematic quantities under a superposed rigid body motion

We now consider other kinematical quantities under the superposed rigid body motion, (3.7). The velocity vector $\underline{v}$, transforms as,

$$\underline{v}^+ \;=\; \frac{\partial \underline{\chi}^+(\underline{X}, t)}{\partial t} \;=\; \frac{d\underline{c}(t)}{dt} \;+\; \frac{d\underline{Q}}{dt}\underline{x} \;+\; \underline{Q}\underline{v}. \tag{3.10}$$

We define $\underline{\Omega}$ as

$$\underline{\Omega} \;=\; \frac{d\underline{Q}}{dt}\underline{Q}^T. \tag{3.11}$$

Recall, that $\underline{QQ}^T = \underline{I}$, so

$$\frac{d\underline{Q}}{dt}\underline{Q}^T \;+\; \underline{Q}\frac{d\underline{Q}^T}{dt} \;=\; 0 \tag{3.12}$$

and hence, using (3.11) and (3.12), we find,

$$\underline{\Omega} + \underline{\Omega}^T \;=\; 0 \tag{3.13}$$

and therefore, $\underline{\Omega}$ is skew-symmetric. Using the definition of Ω, (3.11), we can write the expression for $\underline{v}^+$ as,

$$\underline{v}^+ \;=\; \frac{d\underline{c}(t)}{dt} \;+\; \underline{\Omega}\underline{Q}\underline{x} \;+\; \underline{Q}\underline{v}, \tag{3.14}$$

and

$$v_i^+ \;=\; \frac{dc_i(t)}{dt} \;+\; \Omega_{il}Q_{lm}x_m \;+\; Q_{il}v_l. \tag{3.15}$$

For future use, we now derive the relationship between $\underline{L}^+$ and $\underline{L}$,

$$\begin{aligned} L_{ij}^+ \;=\; \frac{\partial v_i^+}{\partial x_j^+} \;&=\; \frac{\partial v_i^+}{\partial x_k}\frac{\partial x_k}{\partial x_j^+} \\ &=\; \left[\Omega_{il}Q_{lm}\delta_{mk} \;+\; Q_{il}\frac{\partial v_l}{\partial x_k}\right]\frac{\partial x_k}{\partial x_j^+} \\ &=\; \left[\Omega_{il}Q_{lk} \;+\; Q_{il}L_{lk}\right]Q_{jk} \\ &=\; \Omega_{ij} \;+\; Q_{il}L_{lk}Q_{jk} \end{aligned} \tag{3.16}$$

or in coordinate free notation,

$$\underline{L}^+ \;=\; \underline{\Omega} \;+\; \underline{Q}\underline{L}\underline{Q}^T. \tag{3.17}$$

Recalling (1.21) and (3.5), it is clear that

$$\underline{F}^+ \;=\; \frac{\partial \underline{\chi}^+(\underline{X},t)}{\partial \underline{X}}, \;\text{ or }\; F_{iA}^+ \;=\; \frac{\partial \chi_i^+(\underline{X},t)}{\partial X_A}, \tag{3.18}$$

Using (3.18) and (3.7),

$$\begin{aligned} F_{iA}^+ \;&=\; \frac{\partial \hat{\chi}_i^+(\underline{x},t)}{\partial x_j}\,\frac{\partial x_j}{\partial X_A}, \\ &=\; Q_{ij}\,F_{jA} \end{aligned} \tag{3.19}$$

and therefore,

$$\underline{F}^+ \;=\; \underline{Q}\underline{F}. \tag{3.20}$$

Using (3.20) it can also be shown that,

$$J^+ \;=\; J. \tag{3.21}$$

Additionally, the density ρ and the normal to a surface $\underline{n}$, transform as,

$$\rho^+ = \rho, \qquad \underline{n}^+ = \underline{Q}\underline{n}. \tag{3.22}$$

A scaler, vector or second order tensor which transform as $(3.22)_1$, $(3.22)_2$ and $(\textbf{??})_1$, respectively are called **objective**.

Exercise 3.1.1 *Since the superposed rigid body motions preserve distances between arbitrary pairs of material points, they must preserve angles between any two vectors. Starting with the form (3.7) show that this is true.*

Exercise 3.1.2 *Using* $\underline{F}^+ = \underline{Q}\underline{F}$*, show that*

$$\begin{array}{llll} \underline{U}^+ = \underline{U}, & \underline{R}^+ = \underline{Q}\underline{R}, & \underline{V}^+ = \underline{Q}\underline{V}\underline{Q}^T & \\ \underline{C}^+ = \underline{C}, & \underline{E}^+ = \underline{E}, & \underline{b}^+ = \underline{Q}\underline{b}\underline{Q}^T, & \underline{e}^+ = \underline{Q}\underline{e}\underline{Q}^T, \end{array}$$

Exercise 3.1.3 *Using (3.17), it can be shown that*

$$\underline{D}^+ = \underline{Q}\underline{D}\underline{Q}^T, \qquad \underline{W}^+ = \underline{\Omega} + \underline{Q}\underline{W}\underline{Q}^T.$$

The stress vector and stress tensor under a superposed rigid body motion

In this section, the relationship between $\underline{t}^+$ and $\underline{t}$ and the relationship between $\underline{T}^+$ and $\underline{T}$ are considered. Recall that,

$$\underline{t} = \underline{t}(\underline{x}, t; \underline{n}) = \underline{T}(\underline{x}, t)\underline{n}. \tag{3.23}$$

Therefore,

$$\underline{t}^+ = \underline{t}^+(\underline{x}^+, t; \underline{n}^+) = \underline{T}^+(\underline{x}^+, t)\underline{n}^+. \tag{3.24}$$

Recall the relationship between the outward unit normal $\underline{n}$ to a material surface in the current configuration and the corresponding outward unit normal in the superposed motion, $\underline{n}^+ = \underline{Q}\underline{n}$. From this relationship and the linear dependence of the stress vector on $\underline{n}$, we might expect (i) $\underline{t}^+$ and $\underline{t}$ to have the same magnitude and (ii) $\underline{t}^+$ and $\underline{t}$ to have the same orientation relative to $\underline{n}^+$ and $\underline{n}$, respectively. Based on these expectations, we introduce the following assumption

$$\underline{t}^+ = \underline{Q}\underline{t}. \tag{3.25}$$

Using (3.25) and (3.22), it can be shown that,

$$|\underline{t}^+| = |\underline{t}|, \qquad \underline{t}^+ \cdot \underline{n}^+ = \underline{t} \cdot \underline{n}. \tag{3.26}$$

In addition,

$$\underline{t}^+ \;=\; \underline{T}^+\underline{n}^+ \;=\; \underline{T}^+\underline{Q}\underline{n}. \qquad t_i^+ \;=\; T_{ij}^+ n_j^+ \;=\; =\; T_{ij}^+ Q_{jk} n_k. \tag{3.27}$$

Using the assumption (3.25) and (3.4),

$$\underline{t}^+ \;=\; \underline{Q}\underline{t} \;=\; \underline{Q}\underline{T}\underline{n}, \quad \text{and} \quad t_i^+ \;=\; Q_{ij} t_j \;=\; Q_{ij} T_{jk} n_k. \tag{3.28}$$

Combining (3.27) and (3.28), we have that

$$\left(\underline{T}^+\underline{Q} - \underline{Q}\underline{T}\right)\underline{n} \;=\; 0, \quad \text{and} \quad \left(T_{ij}^+ Q_{jk} - Q_{ij} T_{jk}\right) n_k \;=\; 0. \tag{3.29}$$

Since (3.29) must hold for all $\underline{n}$ and the expression in brackets is independent of $\underline{n}$, we can conclude that the expression in brackets is equal to zero. Hence,

$$\begin{aligned} 0 &= \underline{T}^+\underline{Q} - \underline{Q}\underline{T} \\ &= \underline{T}^+\underline{Q}\underline{Q}^T - \underline{Q}\underline{T}\underline{Q}^T \\ &= \underline{T}^+ - \underline{Q}\underline{T}\underline{Q}^T \end{aligned} \quad \text{or} \quad \left(\begin{aligned} 0 &= T_{ij}^+ Q_{jk} - Q_{ij} T_{jk} \\ &= T_{ij}^+ Q_{jk} Q_{lk} - Q_{ij} T_{jk} Q_{lk} \\ &= T_{il}^+ - Q_{ij} T_{jk} Q_{lk} \end{aligned}\right. \tag{3.30}$$

and therefore,

$$\underline{T}^+ \;=\; \underline{Q}\underline{T}\underline{Q}^T, \qquad \text{and} \qquad T_{il}^+ \;=\; Q_{ij} T_{jk} Q_{lk}. \tag{3.31}$$

Exercise 3.1.4 *Prove the identities in equation (3.26).*

3.2 Nonlinear Elastic Material (Cauchy-Elastic Material)

An elastic material in an isothermal theory is characterized by a constitutive equation relating the stress tensor $\underline{T}$ to the deformation gradient $\underline{F}$, namely

$$\underline{T}(\underline{X}, t) = \mathcal{G}(\underline{F}(\underline{X}, t), \underline{X}) \tag{3.32}$$

where $\mathcal{G}$ is a tensor-valued symmetric tensor function called the *response function*. Note that Cauchy elastic materials do not depend on the history of the material, but solely on the current state of deformation. For homogeneous bodies, the material behavior is independent of material point. Thus, we would require that the density ρ_0 be independent of $\underline{X}$ and in addition, the manner in which $\mathcal{G}$ depends on $\underline{F}$ be independent of $\underline{X}$. Of course, the deformation need not be homogeneous, in which case, $\underline{T}$ will indirectly depend on $\underline{X}$,

$$\underline{T} = \mathcal{G}(\underline{F}(\underline{X}, t)). \tag{3.33}$$

3.2.1 Implications of invariance under a superposed rigid body motion

We will now show, that by requiring Cauchy-elastic materials be invariant under superposed rigid body motions, we restrict the possible forms for G($\underline{F}$($\underline{X}$,t),$\underline{X}$). In particular, requiring invariance under a superposed rigid body motion means the form of the stress response is the same for the original motion as well as the superposed motion. Therefore, given the form (3.33) for the original motion, then the response for the superposed motion must be,

$$\underline{T}^+ \; = \; \mathcal{G}(\underline{F}^+, \underline{X}) \qquad \text{for all proper orthogonal } \underline{Q}. \tag{3.34}$$

Then using (3.31) with (3.34), we require

$$\underline{Q}\underline{T}\underline{Q}^T \; = \; \mathcal{G}(\underline{Q}\underline{F}, \underline{X}) \qquad \text{for all proper orthogonal } Q. \tag{3.35}$$

or

$$\underline{T} \; = \; \underline{Q}^T \mathcal{G}(\underline{Q}\underline{F}, \underline{X})\underline{Q} \qquad \text{for all proper orthogonal } Q. \tag{3.36}$$

Without loss in generality, we can choose $\underline{Q} = \underline{R}^T$, where $\underline{F} = \underline{R}\underline{U}$ and $\underline{R}$ is proper orthogonal. We then have from (3.36),

$$\underline{T} \; = \; \underline{R}\mathcal{G}(\underline{R}\underline{R}^T\underline{U}, \underline{X})\underline{R}^T \tag{3.37}$$

or

Restrictions on Cauchy-Elasticity tensor due to invariance requirements

$$\underline{T} \; = \; \underline{R}\,\mathcal{G}(\underline{U}, \underline{X})\,\underline{R}^T. \tag{3.38}$$

We therefore see, that the dependence of $\underline{T}$ on $\underline{R}$ is determined a priori whereas the the dependence of $\underline{T}$ on the stretch tensor $\underline{U}$ is arbitrary. We shall see that there is a large class of elastic materials for which such a dependence must be of a special form. One example is isotropic materials.

3.3 Viscous fluids

In this section, we consider viscous fluids. Namely, the constitutive response of the material depends only on the fluid mass density and the velocity gradient $\underline{L}$ is of the form $\underline{T} = \hat{\underline{T}}(\rho, \underline{L})$. We will now show that if we impose invariance requirements that viscous fluids *must be* isotropic.

Using the relationship

$$\underline{L} \; = \; \underline{W} \; + \; \underline{D} \tag{3.39}$$

we can rewrite the constitutive assumption as

$$\underline{T} \; = \; \hat{\underline{T}}(\rho, \underline{D}, \underline{W}). \tag{3.40}$$

If the constitutive equation (3.40) is a valid one, then it must hold for all motions, in particular, it must hold for a superposed rigid motion,

$$\underline{T}^{+} \;=\; \underline{\hat{T}}(\rho^{+}, \underline{D}^{+}, \underline{W}^{+}). \tag{3.41}$$

Recalling that,

$$\underline{T}^{+} \;=\; \underline{Q} \cdot \underline{T} \cdot \underline{Q}^{T}, \tag{3.42}$$

and using (3.41)

$$\underline{\hat{T}}(\rho^{+}, \underline{D}^{+}, \underline{W}^{+}) \;=\; \underline{Q} \cdot \underline{\hat{T}}(\rho, \underline{D}, \underline{W}) \cdot \underline{Q}^{T}. \tag{3.43}$$

Recalling,

$$\rho^{+} \;=\; \rho, \qquad \underline{D}^{+} \;=\; \underline{Q} \cdot \underline{D} \cdot \underline{Q}^{T}, \qquad \underline{W}^{+} \;=\; \underline{Q} \cdot \underline{W} \cdot \underline{Q}^{T} \;+\; \underline{\Omega}, \tag{3.44}$$

we can write (3.43) as

$$\underline{\hat{T}}(\rho^{+}, \underline{Q} \cdot \underline{D} \cdot \underline{Q}^{T}, \underline{Q} \cdot \underline{W} \cdot \underline{Q}^{T} + \underline{\Omega}) \;=\; \underline{Q} \cdot \underline{\hat{T}}(\rho, \underline{D}, \underline{W}) \cdot \underline{Q}^{T}. \tag{3.45}$$

Now consider a superposed motion, for which at some fixed time t, $\underline{Q} \;=\; \underline{I}$, and $\underline{\Omega} \neq 0$. In this case, (3.45) becomes,

$$\underline{\hat{T}}(\rho, \underline{D}, \underline{W} + \underline{\Omega}) \;=\; \underline{\hat{T}}(\rho, \underline{D}, \underline{W}), \tag{3.46}$$

for arbitrary choices of the skew-symmetric tensor $\underline{\Omega}$. It then follows that the response function $\underline{T}$ cannot depend on $\underline{W}$, and so (3.40) is reduced to

$$\underline{T} \;=\; \underline{\hat{T}}(\rho, \underline{D}), \tag{3.47}$$

and must satisfy

$$\underline{\hat{T}}(\rho, \underline{Q} \cdot \underline{D} \cdot \underline{Q}^{T}) \;=\; \underline{Q} \cdot \underline{\hat{T}}(\rho, \underline{D}) \cdot \underline{Q}^{T}, \tag{3.48}$$

for all proper orthogonal $\underline{Q}$. Then, from (3.48), we can conclude that $\underline{\hat{T}}(\rho, \underline{D})$ must be a symmetric isotropic tensor function of $\underline{D}$, (see, for example, Appendix **??**). A representation theorem for tensors (Appendix **??**), states that the most general second order isotropic tensor function $\underline{\hat{T}}(\rho, \underline{D})$ for symmetric second order tensor $\underline{D}$, has the form

$$\underline{T} \;=\; \phi_0 \, \underline{I} \;+\; \phi_1 \underline{D} \;+\; \phi_2 \underline{D}^2, \tag{3.49}$$

where in general ϕ_0, ϕ_1, ϕ_2 are functions of the three principal invariants of $\underline{D}$ as well as the mass density ρ. The three principal invariants of $\underline{D}$ are,

$$\begin{aligned} I &= \text{tr}\underline{D} \;=\; \text{D}_{\text{ii}} \\ II &= \frac{1}{2}\left(\text{tr}(\underline{D})^2 - \text{tr}(\underline{D}\cdot\underline{D})\right) \;=\; \frac{1}{2}\left(D_{ii}D_{jj} - D_{ij}D_{ji}\right) \\ III &= \det\underline{D}. \end{aligned} \tag{3.50}$$

3.3.1 Compressible Newtonian Fluid

In the special case of (3.49) for which the dependence of $\underline{T}$ on $\underline{D}$ is linear, we recover the compressible Newtonian fluid. It is typically written as,

Compressible Newtonian Fluid

$$\underline{T} \;=\; -p\,\underline{I} \;+\; \lambda\,\text{tr}(\underline{D})\,\underline{I} \;+\; 2\mu\underline{D} \tag{3.51}$$

where p, λ, μ are functions of ρ and λ and μ are viscosity coefficients.

3.3.2 Ideal Fluid

If instead of starting with $\underline{T} \;=\; \hat{\underline{T}}(\rho, \underline{L})$, we had started with $\underline{T} \;=\; \hat{\underline{T}}(\rho)$, we would have obtained from invariance, that

$$\hat{\underline{T}}(\rho) \;=\; \underline{Q}\cdot\hat{\underline{T}}(\rho)\cdot\underline{Q}^T, \tag{3.52}$$

for all proper orthogonal $\underline{Q}$. Therefore, $\underline{T}$ is an isotropic tensor. Theorem 3 of Appendix B gives us that the most general form for $\underline{T}$ is,

Ideal Fluid - compressible

$$\underline{T} \;=\; -p(\rho)\underline{I} \qquad \text{where } p \text{ is a function of } \rho. \tag{3.53}$$

3.4 Hyperelastic Materials

We have already defined a Cauchy elastic body as one in which the Cauchy stress tensor is a function of the deformation tensor. We now define a hyperelastic material for a purely mechanical theory (in which we do not need to introduce thermodynamic considerations such as the balance of energy and Clausius-Duhem inequality). In particular, we will assume that for hyperelastic materials, the stress power can be represented in the following way

$$\underline{T}:\underline{D} \;=\; \rho\,\frac{D\Sigma}{Dt} \qquad \text{or} \qquad T_{ij}\,D_{ij} \;=\; \rho\,\frac{D\Sigma}{Dt}. \tag{3.54}$$

where $\Sigma(\underline{X}, t)$ is called the **strain energy density** function or the **stored energy per unit mass**. In other words, the change in strain energy per unit mass of the body arises from work done on the body by internal stresses. Note, that such a representation is not true for general materials. The total strain energy per unit mass of the body will be denoted by $\mathcal{U}$ and is therefore,

$$\mathcal{U} \;=\; \int_{\mathcal{V}(t)} \rho\,\Sigma\,dv. \tag{3.55}$$

After integrating (3.54) over an arbitrary volume $\mathcal{V}(t)$ and using (2.52) with (3.55), it follows that,

$$\frac{d}{dt}\left(\mathcal{K} \,+\, \mathcal{U}\right) \;=\; \int_{\partial\mathcal{V}(t)} \underline{t}\cdot\underline{v}da \;+\; \int_{\mathcal{V}(t)} \rho\underline{b}\cdot\underline{v}dv, \tag{3.56}$$

where $\mathcal{K}$ is the kinetic energy in the body that was defined earlier. We see that for hyperlastic materials, work on the body is directly converted to kinetic or stored energy. Equation (3.56) is the statement that the rate of change of kinetic energy plus the rate of change of strain energy equals the rate of work by surface and body forces.

In classical hyperelasticity, it is assumed that the strain energy at each material point and for all time depends on the deformation gradient at the same point and time,

$$\Sigma \;=\; \tilde{\Sigma}(\underline{F},\underline{X}). \tag{3.57}$$

For homogeneous materials, the form of the function given in (3.57) will be the same at all points $\Sigma \;=\; \tilde{\Sigma}(\underline{F})$. A normalization condition is typically applied to the strain energy function, so that the strain energy vanishes in the reference configuration where $\underline{F} \;=\; \underline{I}$,

$$\tilde{\Sigma}(\underline{I}) \;=\; 0. \tag{3.58}$$

Often a strain energy per unit volume in the *reference* configuration W is introduced,

$$\mathcal{U} \;=\; \int_{\mathcal{V}_o} W\,dV. \tag{3.59}$$

and therefore, from (3.55), it follows that, $W = \rho\,J\,\Sigma \;=\; \rho_o\,\Sigma$.

Work done on a hyperelastic material in a closed dynamical process

We now evaluate the work done on a hyperelastic material in a closed dynamic process by the internal stress field. We first consider a dynamical process defined by the pair $\underline{T}$ and $\underline{x} \;=\; \underline{\chi}(\underline{X},t)$ which takes place during the time interval $t \in [t_1,t_2]$. A dynamical process is closed if $\underline{F}(\underline{X},t_1) = \underline{F}(\underline{X},t_2)$ for all $\underline{X} \subseteq \mathcal{R}_0$. Now consider the work done on a hyperelastic material as defined by (3.54) at arbitrary material point $\underline{X}$ during a closed dynamical process,

$$\text{Work done during } t \in [t_1,t_2] \qquad \begin{aligned} \int_{t_1}^{t_2} \underline{T}:\underline{D}dt \;&=\; \int_{t_1}^{t_2} \rho\frac{\partial\Sigma(\underline{F}(\underline{X},t))}{\partial t}dt \\ &=\; \rho\left(\Sigma(\underline{F}(\underline{X},t_2)) \;-\; \Sigma(\underline{F}(\underline{X},t_1))\right) \\ &=\; 0. \end{aligned} \tag{3.60}$$

Therefore, the work done by the stress field on a hyperelastic material is zero during a closed dynamical process, independent of the deformation (path) during the interval between t_1 and t_2

3.4.1 Relationship between W and the Helmholtz free energy

If we extend our approach to include the concepts introduced above for thermodynamics, we can relate the strain energy density and strain energy per unit volume to thermodynamic entities,

$$\begin{array}{ll} \textbf{Isothermal processes } (\dot{T}=0) & \\ W = \rho_0\,\psi, & \Sigma = \psi \\ & \\ \textbf{Isentropic processes, } (\dot{\eta}=0) & \\ W = \rho_0 u, & \Sigma = u \end{array} \tag{3.61}$$

The details of these results are not shown here, but can be found, for example in [27].

3.4.2 Invariance restrictions

Invariance requirements restrict the form of the functional dependence of the strain energy function on $\underline{F}$. It follows from the relationship between the strain energy function and the stress tensor, and our invariance requirements on the stress tensor, that

$$W(\underline{F}) = W(\underline{QF}), \qquad \text{for all proper orthogonal } \underline{Q}. \tag{3.62}$$

It therefore must hold for $\underline{Q} = \underline{R}$, where $\underline{F} = \underline{RU}$ and

$$W(\underline{F}) = W(\underline{U}). \tag{3.63}$$

Without loss in generality, the most general form of the strain energy function that satisfies invariance can be written as, $W = W(\underline{C})$. It is straightforward to show that this is also a *sufficient condition* for the strain energy to satisfy invariance requirements.

Any function of $\underline{U}$ can also be written as a function of $\underline{C}$ or $\underline{E}$. In particular, it follows from (3.54) and the chain rule that,

$$\underline{T} = \frac{1}{J}\underline{F}\left(\frac{\partial W}{\partial \underline{C}} + \frac{\partial W}{\partial \underline{C}^T}\right)\underline{F}^T \qquad T_{ij} = \frac{1}{J}F_{iA}F_{jB}\left(\frac{\partial W}{\partial C_{AB}} + \frac{\partial W}{\partial C_{BA}}\right)$$

$$\underline{P} = \underline{F}\left(\frac{\partial W}{\partial \underline{C}} + \frac{\partial W}{\partial \underline{C}^T}\right) \qquad \text{or} \qquad P_{iB} = F_{iA}\left(\frac{\partial W}{\partial C_{AB}} + \frac{\partial W}{\partial C_{BA}}\right)$$

$$\underline{S} = \left(\frac{\partial W}{\partial \underline{C}} + \frac{\partial W}{\partial \underline{C}^T}\right) \qquad S_{AB} = \left(\frac{\partial W}{\partial C_{AB}} + \frac{\partial W}{\partial C_{BA}}\right)$$

where $\hat{W}(\underline{C})$ must be a symmetric function of $\underline{C}$.

(3.64)

We see that the stress field $\underline{S}$ in a hyperelastic material is completely determined by a scalar function of $\underline{C}$. In some works, rather than (3.64) the following equations are used,

$$\underline{T} = 2\frac{1}{J}\underline{F}\frac{\partial W}{\partial \underline{C}}\underline{F}^T, \qquad \underline{P} = 2\underline{F}\frac{\partial W}{\partial \underline{C}}, \quad \underline{S} = 2\frac{\partial W}{\partial \underline{C}}. \tag{3.65}$$

As detailed on page 212-213 of [27], care must be taken in using (3.65). Briefly, consider the example $W = c(C_{12}^2 + C_{21}^2)$, which can equivalently be written as $W = 2cC_{12}^2$. If we calculate S_{12} and S_{21} using (3.64), we obtain the same result for both representations of W. However, if we use (3.65), we obtain $S_{12} = S_{21} = 2(C_{21} + C_{21})$ using the first representaton (correct), but $S_{12} = 8C_{21}, S_{21} = 0$ (incorrect), using the second. It is therefore, in some sense, safer to use (3.64).

3.5 Materials with Symmetry

In this section, we will discuss material symmetry of a body. Some useful references on this subject are [26, 31], Truesdell and Noll [1992,Section 33] [30] Ogden [1997,Section 4.2.5][20], Spencer [1980, Section 8.2] [25].

Consider an arbitrary material point in the body P, identified by its location in the reference configuration $\underline{X}$. In order to discuss the symmetry groups of the body, we need to consider possible changes to the response of a body when the body is rigidly rotated and possibly translated in the reference configuration. Under this rigid motion, the material point P will moved from $\underline{X}$ to $\underline{X}^*$ by a translation vector $\underline{c}$ and rotated using an orthogonal tensor $\underline{A}$ [2],

$$\underline{X}^* = \underline{c} + \underline{A}\cdot\underline{X}. \tag{3.66}$$

[2] $\underline{A}$ is not necessarily proper orthogonal, meaning we allow reflections

We describe the subsequent motion of the body relative to its position in reference configuration κ_0 or κ_0^* as,

$$\underline{x} = \underline{\chi}(\underline{X}, t) = \underline{\chi}^*(\underline{X}^*, t). \tag{3.67}$$

Using the chain rule with (3.66) and (3.67), we obtain a relation between $\underline{F}$ and $\underline{F}^*$,

$$F_{iA} = \frac{\partial \chi_i^*(\underline{X}^*, t)}{\partial X_B^*} \frac{\partial X_B^*}{\partial X_A} = F_{iB}^* A_{BA} \tag{3.68}$$

or

$$\underline{F}^* = \underline{F} \cdot \underline{A}^T. \tag{3.69}$$

It therefore follows that,

$$\underline{C}^* = \underline{A}\,\underline{C}\,\underline{A}^T \quad \text{and} \quad \underline{b}^* = \underline{b}. \tag{3.70}$$

We now consider the change in the response of the body to particular rigid rotations and reflections of the reference configuration, (different choices of $\underline{A}$). In particular, we will describe symmetries of the body, based on the invariance of their response to rotations in κ_0, $W(\underline{C}) = W(\underline{C}^*)$. Namely, the symmetry of the body will be stated in terms of the set of $\underline{A}$ for which

$$W(\underline{C}) = W(\underline{A}\,\underline{C}\,\underline{A}^T). \tag{3.71}$$

3.5.1 Isotropic hyperelastic materials

For isotropic materials, (3.71) holds for all orthogonal $\underline{A}$ and therefore the symmetry group of an isotropic material includes all rotations about all possible axes and reflections in any plane. Namely, it is the group of all orthogonal tensors (not just proper), which is the full orthogonal group in three dimensions [3].

Isotropic Hyperelastic Material

$$W(\underline{C}) = W(\underline{A}\,\underline{C}\,\underline{A}^T) \text{for all orthogonal } \underline{A} \tag{3.72}$$

From a practical point of view, material isotropy means that if we load a mass of this material which was spherical in the unloaded reference configuration, the response of the material will be unaffected by how we rotated or reflected this sphere in κ_0.

Note that the mathematical requirement that (3.71) holds for all orthogonal $\underline{A}$ is a very different requirement than that stated for invariance under a super posed rigid body motion. The invariance requirement must be satisfied for all materials, whereas the condition (3.71) is met by some materials for a particular choice of $\underline{A}$ and is a material property. In addition, we only require that invariance holds for all proper orthogonal tensors, $\underline{Q}$, whereas $\underline{A}$ includes orthogonal tensors which are not proper.

[3] A material whose symmetry group contains all rotations but no reflections is called *Hemitropic*. For many purposes, this distinction is not important

Mathematically, (3.72) implies that $W(\underline{C})$ is a scalar valued isotropic tensor function of $\underline{C}$. It then follows from a representation theorem for isotropic scalar functions (see, Appendix D), that without loss in generality, the strain energy function for an isotropic, hyperelastic material may be expressed as a function of the scalar invariants of $\underline{C}$, denoted as I_1, I_2, I_3,

$$W \;=\; W(I_1, I_2, I_3), \tag{3.73}$$

where

$$\begin{aligned} I_1 &= \mathrm{tr}\underline{C} \;=\; \mathrm{C_{AA}} \\ I_2 &= \frac{1}{2}\left(\mathrm{tr}(\underline{C})^2 - \mathrm{tr}(\underline{C}\cdot\underline{C})\right) \;=\; \frac{1}{2}\left(C_{AA}C_{BB} - C_{AB}C_{AB}\right) \\ I_3 &= \det\underline{C}. \end{aligned} \tag{3.74}$$

Since $\underline{B}$ and $\underline{C}$ have the same eigenvalues, their invariants can be shown to be identical.

Stress tensors for an isotropic, elastic material

Using the chain rule with (3.64), (3.73) and results from Appendix D,

$$\begin{aligned} \frac{\partial W}{\partial C_{AB}} &= \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial C_{AB}} \;+\; \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial C_{AB}} \;+\; \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial C_{AB}} \\ &= \frac{\partial W}{\partial I_1}\delta_{AB} \;+\; \frac{\partial W}{\partial I_2}(I_1\delta_{AB} - C_{AB}) \;+\; \frac{\partial W}{\partial I_3}(I_3 C^{-1}_{AB}) \end{aligned} \tag{3.75}$$

Hence,

$$\underline{T} \;=\; \frac{2}{J}\underline{F}\left(\frac{\partial W}{\partial I_1}\underline{I} \;+\; \frac{\partial W}{\partial I_2}(I_1\underline{I} - \underline{C}) \;+\; \frac{\partial W}{\partial I_3}(I_3\underline{C}^{-1})\right)\underline{F}^T. \tag{3.76}$$

Recalling that $\underline{S} \;=\; J\underline{F}^{-1}\underline{T}\,\underline{F}^{-T}$, were therefore have,

For an isotropic elastic material

$$\underline{T} \;=\; 2\left[I_3^{1/2}W_3\,\underline{I} \;+\; I_3^{-1/2}(W_1 + I_1W_2)\,\underline{b} \;-\; I_3^{-1/2}W_2\,\underline{b}^2\right]$$

and

$$\underline{S} \;=\; 2\left[(W_1 + I_1\,W_2)\,\underline{I} \;-\; W_2\,\underline{C} \;+\; W_3\,I_3\,\underline{C}^{-1}\right] \tag{3.77}$$

where $W_i \equiv \partial W/\partial I_i$.

In some cases, it may be of interest to write (3.77) with respect to different powers of the stress tensor. This can easily be done using results from the Cayley-Hamilton Theorem. In particular, we can use the result that a second order tensor $\underline{G}$ satisfies its own characteristic equation so that,

$$\underline{G}^3 \;-\; I_1\underline{G}^2 \;+\; I_2\underline{G} \;-\; I_3\underline{I} \;=\; 0, \tag{3.78}$$

or

$$\underline{G}^2 - I_1\underline{G} + I_2\underline{I} - I_3\underline{G}^{-1} = 0. \tag{3.79}$$

Exercise: It is left as an exercise to use the polar decomposition theorem with $\underline{A} = \underline{R}(\underline{X}, t)$ to shown that if an elastic material with $W = W(\underline{C})$ is isotropic, then equivalently $W = W(\underline{b})$. This last result is also clear from (3.73)

Exercise: Show that $W(I_1^*, I_2^*, I_3^*) = W(I_1, I_2, I_3)$ for all orthogonal $\underline{A}$, by first showing that $I_1^* = I_1$, $I_2^* = I_2$, $I_3^* = I_3$ for all orthogonal tensors $\underline{A}$, where it should be recalled that $\underline{C}^* = \underline{ACA}^T$. Namely, show that I_1, I_2, I_3 are invariants of $\underline{C}$.

Representative compressible, isotropic strain energy functions

For compressible, isotropic materials, the strain energy may depend on all three invariants. A very general strain energy function for compressible materials is,

$$W = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} c_{ijk}(I_1 - 3)^i\,(I_2 - 3)^j\,(I_3 - 3)^k \tag{3.80}$$

where c_{ijk} are material constants. There are a number of other well known compressible strain energy functions, see, e.g. [27].

Incompressible, isotropic strain energy functions

For incompressible material, I_3 is a constant, so the strain energy function will only be a function of the first two invariants. In addition to the work part of the tensor, there will be a Lagrange multiplier (denoted as p, but not necessarily equivalent to the mechanical pressure). Some commonly used isotropic strain energy functions for biological materials include the (i) Mooney-Rivlin, (ii) Neo-Hookean and (iii) exponential strain energy functions. The **Mooney-Rivlin** strain energy function has frequently been used for rubbers and can be written as

$$W = \frac{\alpha}{2}(I_1 - 3) + \frac{\beta}{2}(I_2 - 3) \tag{3.81}$$

where α and β are material constants. Notice that W is normalized such that for $\underline{F} = \underline{I}$, $W = 0$. The **neo-Hookean material** is a special Mooney-Rivlin material for which $\beta = 0$.

A special type of hyperelastic material with an **exponential** dependence on the first invariant of $\underline{b}$ was introduced by Fung to model the nonlinearly elastic response of biological tissue, [**?**],

$$W(I_1) = \frac{\alpha}{2\gamma}\left(e^{\gamma(I_1-3)} - 1\right), \tag{3.82}$$

where α and γ are material constants. This strain energy has been generalized to include a dependence on both I_1 and I_2

$$W(I_1, I_2) = C\left(e^{b_1(I_1-3)+b_2(I_2-3)} - 1\right). \tag{3.83}$$

The components of the Cauchy stress tensor for an isotropic, incompressible hyperelastic material can then be reduced from (3.65) to,

$$\underline{T} = -p\underline{I} + 2\,\rho_0\,\frac{\partial \Sigma}{\partial I_1}\underline{b} - 2\,\rho_0\,\frac{\partial W}{\partial I_2}\underline{b}^{-1}, \tag{3.84}$$

where p is the Lagrange multiplier arising from incompressibility. Alternatively, using $W = \rho_0 \Sigma$,

$$\underline{T} = -p\underline{I} + 2\,\frac{\partial W}{\partial I_1}\underline{b} - 2\,\frac{\partial W}{\partial I_2}\underline{b}^{-1}. \tag{3.85}$$

It follows from (3.85) that the corresponding Cauchy stress tensors for these materials are, for the Mooney-Rivlin material is,

$$\begin{array}{llcl}
\text{Neo-Hookean} & \underline{T} & = & -p\underline{I} \,+\, \alpha\underline{b} \\
\text{Mooney-Rivlin} & \underline{T} & = & -p\underline{I} \,+\, \alpha\underline{b} \,-\, \beta\underline{b}^{-1} \\
\text{1-term Exponential} & \underline{T} & = & -p\underline{I} \,+\, \alpha e^{\gamma(I_1-3)}\,\underline{b} \\
\text{2-term Exponential} & \underline{T} & = & -p\underline{I} \,+\, 2\,C(b_1\,e^{b_1(I_1-3)}\,\underline{b} \,-\, b_2\,e^{b_2(I_1-3)}\,\underline{b}^{-1})
\end{array} \tag{3.86}$$

where it should be recalld that $\alpha, \beta, \gamma, b_1$ and b_2 are constants.

3.5.2 Transversely Isotropic Materials

A transversely isotropic material has a single axis of symmetry for each material point in the undeformed configuration. If we select a local Cartesian coordinate system at point P, we can denote this axis as the Z_3 axis. The material is then isotropic in the Z_1Z_2 plane. This is equivalent to saying the symmetry group contains all rotations about the Z_3 axis. With respect to this choice of coordinates, we can therefore write $\underline{A}$ as

$$[\underline{A}] \;=\; \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.87}$$

with $\theta \in [0, 2\pi]$. The strain energy for materials which are transversely isotropic must be invariant to any rotation in the Z_1Z_2 plane. Which is equivalent to stating (3.71) holds for all $\underline{A}$ of the form (3.87). For example, a transversely isotropic material might be a composite material composed of bundle of identical fibers embedded in an isotropic matrix and aligned in the Z_3 direction. Based on these symmetry requirements, it can be shown that, without loss in generality, that a transversely isotropic material can be written as,

Transversely Isotropic Hyperelastic Material

$$W \;=\; W(I_1, I_2, I_3, I_4, I_5) \tag{3.88}$$

where

$$I_4 \;=\; \underline{m}\cdot\underline{C}\underline{m} \quad \text{and} \quad I_5 \;=\; \underline{m}\cdot\underline{C}^2\underline{m} \;-\; I_4^2,$$

where $\underline{m}$ is a unit vector parallel to the symmetry axis.

Notice, that I_4 is the stretch squared of an infinitesimal element originally aligned with the $\underline{m}_3$ axis. For example, if we choose local coordinate with $\underline{m} = \underline{e}_3$, then $I_4 = C_{33}$, which is the stretch squared of an infinitesimal element originally aligned with $\underline{e}_3$. Furthermore, $I_5 = C_{13}^2 + C_{23}^2$. We can obtain some physical insights about I_5 by recalling from Section 1.7, that

$$\cos\alpha_{13} \;=\; \frac{C_{13}}{\sqrt{C_{11}C_{33}}}, \qquad \cos\alpha_{23} \;=\; \frac{C_{23}}{\sqrt{C_{22}C_{33}}}, \tag{3.89}$$

where α_{13} is the current angle between two infinitesimal elements which were aligned with the $\underline{e}_1$ and $\underline{e}_3$ axis in κ_0. Similarly, α_{23} is the current angle between two infinitesimal elements which were aligned with the $\underline{e}_2$ and $\underline{e}_3$ axis in κ_0.

Examples of biological tissue which are sometimes modeled as transversely isotropic include skeletal muscle and the layers of heart muscle. For example, each layer is sometimes modeled as,

$$W \;=\; b_1 \exp\left(b_2(I_1 - 3)\right) \;+\; b_3(I_4 - 1)^m \tag{3.90}$$

where I_4. The variables b_1, b_2, b_3 are material constants and m is an even integer. In this example, the first term represents the combined effects of an isotropic matrix and the second term represents the fiber contributions.

Orthotropic Materials

A material is orthotropic if for each point of the material in the reference configuration possesses three mutually orthogonal symmetry planes. The orthogonal axis may vary from point to point. An orthotropic material might, for example, be a material composed of orthogonally oriented fibers embedded in an isotropic matrix. Wood is often modeled as orthotropic. Hence, the set of $\underline{A}$ can be written in component form with respect to the a basis with members parallel to each of these directions,

$$[\underline{A}] \;=\; \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{3.91}$$

The strain energy function of an orthotropic material must be unaffected by reflections of each of the three orthogonal axis. This is equivalent to stating that (3.71) must hold for all $\underline{A}$ of the form, (3.91). [4]

3.6 Finite Elastic Constitutive Equation for Fiber-Reinforced Material

3.6.1 Constitutive Equation for One Family of Fibers

The previous discussion of anisotropic materials was purely phenomenological. Namely, we did not directly include information about the constituents of the material. Instead, we

[4] This section is not complete yet

considered the bulk response. In this section, we will take a different approach in which the role of fibers in the material is directly incorporated into the constitutive model. This discussion will closely follow the work of Spencer, [24, 26]. See also, Sections 6.7 and 6.8 of [14] for a summary of the work of Spencer and further comments. For additional applications and background material see [1].

3.6.2 Kinematics

In this section, we consider a body $\mathfrak{B}$ which occupies a reference configuration κ_0 at some time $t = 0$. and As the body deforms in time, it will occupy the configuration $\kappa(t)$ at an arbitrary current time t. This body is assumed to be endowed with fibers in a continuous manner so that every material point identified by $\underline{X}$ in κ_0 is endowed with a family of fibers with orientation $\underline{a}_0$. We assume that during the deformation, fibers will be convected with the underlying material. In general, the fiber will rotate and stretch. For example, if fiber element has direction $\underline{a}_0$ and length ΔX in configuration κ_0, it will have a new direction $\underline{a}$ and new length Δx in the current configuration. For definiteness, we take $\underline{a}_0$ and $\underline{a}$ to be unit vectors.

For example, consider a fiber whose ends are at coordinates, $\underline{X}$ and $\underline{X} + \Delta X \underline{a}_0$ in κ_0 and $\underline{x}$ and $\underline{x} + \Delta x \underline{a}$ in κ. Then we can obtain the relationship between $\underline{a}_0$ and $\underline{a}$ as follows. From (1.1),

$$\underline{x} + \Delta x \underline{a} \; = \; \chi(\underline{X} + \Delta X \underline{a}_0, t). \tag{3.92}$$

If we perform a Taylor series expansion of $\chi(\underline{X} + \Delta X \underline{a}_0)$ about $\underline{X}$,

$$\underline{x} + \Delta x \underline{a} \; = \; \chi(\underline{X}, t) \; + \; \Delta X \underline{F}(\underline{X}, t) \cdot \underline{a}_0 \; + \; H.O.T.. \tag{3.93}$$

Taking the limit as $\Delta \underline{X}$ tends to zero, we find,

$$\underline{a} dx \; = \; \underline{F}(\underline{X}, t) \cdot \underline{a}_0 dX. \tag{3.94}$$

Therefore,

$$\boxed{\lambda \underline{a} \; = \; \underline{F}(\underline{X}, t) \cdot \underline{a}_0, \qquad \lambda \; = \; \frac{dx}{dX}.} \tag{3.95}$$

Since $\underline{a}$ and $\underline{a}_0$ are unit vectors, it then follows that,

$$\lambda^2 \; = \; \underline{a}_0 \cdot \underline{C}(\underline{X}, t)\, \underline{a}_0. \tag{3.96}$$

This should not be surprising. We have already shown that diagonal components of $\underline{C}$ represent the stretch squared of an infinitesimal material element which was aligned with the unit base vectors in the undeformed configuration. We see from (3.96), that the direction of the fiber in $\kappa(t)$ is determined entirely by $\underline{C}$ and the fiber direction in κ_0.

As before, we consider hyperelastic materials in a purely mechanical setting, by which we mean the stress power can be represented as,

$$\underline{T} : \underline{D} \; = \; \rho \frac{D\Sigma}{Dt} \qquad \text{or, equivalently} \qquad \underline{T} : \underline{D} \; = \; = \; \frac{1}{J} \frac{DW}{Dt}. \tag{3.97}$$

Here, we assume the strain energy functions depends on a measure of strain or stretch of the bulk material as well as a fiber direction in the reference configuration. Since a flipping of the fiber end to end should not play a role, we assume the dependence on $\underline{a}_0$ is even, (for example, through $\underline{a}_0 \otimes \underline{a}_0$)[5],

$$W \;=\; \hat{W}(\underline{C}, \underline{a}_0 \otimes \underline{a}_0). \tag{3.98}$$

Note that since $\underline{a}_0$ is independent of time, the relationship between the strain energy function and the stress tensor is that same as for the classical hyperelastic material,

$$\underline{T} \;=\; \frac{1}{J}\,\underline{F}\left(\frac{\partial W}{\partial \underline{C}} \;+\; \frac{\partial W}{\partial \underline{C}^T}\right)\underline{F}^T. \tag{3.99}$$

We will now assume the only anisotropy in the material is due to the fiber orientation as characterized by $\underline{a}_0$. In this case, we will assume that the strain energy is unchanged by any rigid rotation, translation and possible reflection of both the matrix material and the fibers around a preferred direction (axis). Mathematically, this is equivalent to the statement

$$\hat{W}(\underline{C}, \underline{a}_0 \otimes \underline{a}_0) \;=\; \hat{W}(\underline{C}^*, \underline{a}_0^* \otimes \underline{a}_0^*). \tag{3.100}$$

where,

$$\underline{C}^* \;=\; \underline{A} \cdot \underline{C} \cdot \underline{A}^T, \qquad \underline{a}_0^* = \underline{A} \cdot \underline{a}_0 \tag{3.101}$$

for all orthogonal second order tensors $\underline{A}$. Note that $\underline{A}$ does not need to be proper orthogonal because the response is also invariant to reflections. Substituting the results (3.101) in (3.100), we obtain

$$\hat{W}(\underline{C}, \underline{a}_0 \otimes \underline{a}_0) \;=\; \hat{W}(\underline{A} \cdot \underline{C} \cdot \underline{A}^T, \underline{A} \cdot \underline{a}_0 \otimes \underline{a}_0 \cdot \underline{A}^T), \text{for all proper orthogonal } \underline{A}. \tag{3.102}$$

In other words, W is a scalar valued isotropic tensor function of the tensor $\underline{C}$ and the dydadic product $\underline{C}$ and $\underline{a}_0 \otimes \underline{a}_0$. Mathematically, (3.102) implies that without loss in generality the strain energy can be written as a function of the integrity basis of $\underline{C}$ and $\underline{a}_0 \otimes \underline{a}_0$ (see, Appendix X for a proof),

$$\hat{W} \;=\; \tilde{W}(I_1, I_2, I_3, I_4, I_5). \tag{3.103}$$

where

$$\begin{aligned} I_1 \;&=\; \mathrm{tr}\underline{C}, \qquad \mathrm{I}_2 \;=\; \frac{1}{2}\left((\mathrm{tr}\underline{C})^2 - \mathrm{tr}(\underline{C}^2)\right), \qquad \mathrm{I}_3 \;=\; \det \underline{C} \;=\; (\rho_0/\rho)^2, \\ I_4 \;&=\; \underline{a}_0 \cdot (\underline{C}\underline{a}_0) \;=\; \lambda^2, \qquad I_5 \;=\; \underline{a}_0 \cdot (\underline{C}^2\underline{a}_0). \end{aligned} \tag{3.104}$$

[5]Note that we could have started with $W = \hat{W}(\underline{F}, \underline{a}_0 \otimes \underline{a}_0)$ and shown that a necessary condition for invariance is that $\hat{W}(\underline{F}, \underline{a}_0 \otimes \underline{a}_0) = \hat{W}(\underline{U}, \underline{a}_0 \otimes \underline{a}_0)$. It is trivial to show this is also a sufficient condition. Therefore, without loss in generality, a strain energy $W = \hat{W}(\underline{F}, \underline{a}_0 \otimes \underline{a}_0)$ that satisfies invariance requirements can be written as (3.98)

Clearly, we have returned to the same form of the strain energy as we obtained earlier when we required considered transverse isotropy Section 3.5.2. The components of the Cauchy stress tensor can be calculated from (3.99) using (3.103) and (3.104),

$$\begin{aligned} T_{ij} &= \frac{1}{J} F_{iR} F_{jS} \left(\frac{\partial W}{\partial C_{RS}} + \frac{\partial W}{\partial C_{SR}} \right) \\ &= I_3^{-1/2} F_{iR} F_{jS} \sum_{k=1}^{5} \frac{\partial W}{\partial I_k} \left(\frac{\partial I_k}{\partial C_{RS}} + \frac{\partial I_k}{\partial C_{SR}} \right). \end{aligned} \tag{3.105}$$

Using the results in Appendix D for the $\partial I_k / \partial \underline{C}$ and the result (3.95), we can rewrite (3.105) as,

$$\begin{aligned} \underline{T} &= 2\, I_3^{-1/2} \left\{ I_3 \frac{\partial W}{\partial I_3} \underline{I} + \left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) \underline{b} - \frac{\partial W}{\partial I_2} \underline{b}^2 \right. \\ &\left. + I_4 \frac{\partial W}{\partial I_4} \underline{a} \otimes \underline{a} + I_4 \frac{\partial W}{\partial I_5} \left(\underline{a} \otimes \underline{b}\, \underline{a} + \underline{b}\, \underline{a} \otimes \underline{a} \right) \right\}, \end{aligned} \tag{3.106}$$

where it should be recalled $\underline{b} = \underline{F}\underline{F}^T$. Using the Cayley-Hamilton theorem in the form,

$$\underline{b}^2 - I_1 \underline{b} + I_2 \underline{I} - I_3 \underline{b}^{-1} = 0. \tag{3.107}$$

we can remove the dependence of (3.106) on $\underline{b}^2$ and introduce a dependence on $\underline{b}^{-1}$,

$$\begin{aligned} \underline{T} &= 2\, I_3^{-1/2} \left\{ \left(I_2 \frac{\partial W}{\partial I_2} + I_3 \frac{\partial W}{\partial I_3} \right) \underline{I} + \frac{\partial W}{\partial I_1} \underline{b} - I_3 \frac{\partial W}{\partial I_2} \underline{b}^{-1} \right. \\ &\left. + I_4 \frac{\partial W}{\partial I_4} \underline{a} \otimes \underline{a} + I_4 \frac{\partial W}{\partial I_5} \left(\underline{a} \otimes \underline{b}\, \underline{a} + \underline{b}\, \underline{a} \otimes \underline{a} \right) \right\}. \end{aligned} \tag{3.108}$$

At noted by Spencer, this last result is equivalent to that obtained by Ericksen and Rivlin for transversely isotropic elastic materials [5]. The Piola-Kirchoff stress tensor corresponding to (3.106) is (where $\underline{S} = I_3^{1/2} \underline{F}^{-1} \cdot \underline{T} \cdot \underline{F}^{-T}$),

$$\begin{aligned} \underline{S} &= 2 \left\{ \left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) \underline{I} - \left(\frac{\partial W}{\partial I_2} \right) \underline{C} \right. \\ &\left. + I_3 \frac{\partial W}{\partial I_3} \underline{C}^{-1} + \frac{\partial W}{\partial I_4} \underline{a}_0 \otimes \underline{a}_0 + \frac{\partial W}{\partial I_5} \left(\underline{a}_0 \otimes \underline{C}\underline{a}_0 + \underline{C}\underline{a}_0 \otimes \underline{a}_0 \right) \right\}. \end{aligned} \tag{3.109}$$

Incompressible Fiber-Reinforced Material

For incompressible materials, $I_3 = 1$ and a constraint response proportional to $\underline{I}$ must be added to the stress tensor for the compressible material. Without loss in generality, this constraint term can absorb all terms proportional to $\underline{I}$ so the Cauchy stress tensor becomes, so that,

$$\underline{T} = 2 \left\{ \frac{\partial W}{\partial I_1} \underline{b} - \frac{\partial W}{\partial I_2} \underline{b}^{-1} + I_4 \frac{\partial W}{\partial I_4} \underline{a} \otimes \underline{a} + I_4 \frac{\partial W}{\partial I_5} \left(\underline{a} \otimes \underline{b}\, \underline{a} + \underline{b}\, \underline{a} \otimes \underline{a} \right) \right\} - p\underline{I} \tag{3.110}$$

where p plays the role of a Lagrange multiplier.

Incompressible Fiber-Reinforced Material with Inextensible Fibers

If, additionally, the fibers are inextensible and the material is incompressible, then $I_4 = 1$ and a corresponding constraint response that does no work must be added directly to the stress tensor,

$$\underline{T} \;= 2\left\{\frac{\partial W}{\partial I_1}\underline{b} \;-\; \frac{\partial W}{\partial I_2}\underline{b}^{-1} \;+\; \frac{\partial W}{\partial I_5}\left(\underline{a}\otimes\underline{b}\cdot\underline{a} \;+\; \underline{a}\cdot\underline{b}\otimes\underline{a}\right)\right\} \;-\; p\underline{I} \;-\; q\,\underline{a}\otimes\underline{a}. \qquad (3.111)$$

The corresponding second Piola-Kirchoff tensor is,

$$\begin{aligned}\underline{S} \;&= 2\left\{\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\underline{I} \;-\; \frac{\partial W}{\partial I_2}\underline{C}\right.\\ &\left.+\frac{\partial W}{\partial I_5}\left(\underline{a}_0\otimes\underline{C}\,\underline{a}_0 \;+\; \underline{C}\,\underline{a}_0\otimes\underline{a}_0\right)\right\} \;-\; p\underline{C}^{-1} \;-\; q\,\underline{a}_0\otimes\underline{a}_0.\end{aligned} \qquad (3.112)$$

3.6.3 Finite Elasticity for Two Families of Fibers

In this section, we consider the case where the material is characterized by two families of fibers. Namely, there are two distinct orientations of interest, characterized by fiber directions $\underline{a}_0$ and $\underline{g}_0$ in the reference configuration κ_0 and in the deformed configuration at the current time by $\underline{a}$ and $\underline{g}$, respectively.

As before, we consider hyperelastic materials in a purely mechanical setting, by which we mean the stress power can be represented as (3.97). Here, we assume the strain energy functions depends on a measure of strain or stretch of the bulk material as well as two fiber directions in the reference configuration. As for the case of a single family of fibers, the direction of the fibers should not play a role, so we assume the dependence on $\underline{a}_0$ and $\underline{g}_0$ is even,

$$\hat{W} \;=\; \hat{W}(\underline{C}, \underline{a}_0\otimes\underline{a}_0, \underline{g}_0\otimes\underline{g}_0). \qquad (3.113)$$

Once again, we assume the only anisotropy in the material is that arising from the fibers. Hence, the strain energy is unchanged by any rigid rotation of both the matrix material and the fibers around a preferred direction (axis). Following similar arguments as for materials with one family of fibers, we require,

$$\hat{W}(\underline{C}, \underline{a}_0\otimes\underline{a}_0, \underline{g}_0\otimes\underline{g}_0) \;=\; \hat{W}(\underline{A}\cdot\underline{C}\cdot\underline{A}^T, \underline{A}\cdot\underline{a}_0\otimes\underline{a}_0\cdot\underline{A}^T, \underline{A}\cdot\underline{g}_0\otimes\underline{g}_0\cdot\underline{A}^T), \text{for all proper orthogonal } \underline{A}. \qquad (3.114)$$

In other words, W is a scalar valued isotropic tensor function of the three tensors $\underline{C}$,$\underline{a}_0\otimes\underline{a}_0$, and $\underline{g}_0\otimes\underline{g}_0$. Without loss in generality the strain energy can be written as a function of the integrity basis of $\underline{C}$, $\underline{a}_0\otimes\underline{a}_0$ and $\underline{g}_0\otimes\underline{g}_0$ (see, Appendix X for a proof),

$$\hat{W} \;=\; \tilde{W}(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9). \qquad (3.115)$$

where [6]

$$
\begin{aligned}
I_1 &= \mathrm{tr}\underline{C}, \qquad \mathrm{I}_2 = \frac{1}{2}\left((\mathrm{tr}\underline{C})^2 - \mathrm{tr}(\underline{C}^2)\right), \qquad \mathrm{I}_3 = \det \underline{C} = (\rho_0/\rho)^2, \\
I_4 &= \underline{a}_0 \cdot \underline{C}\, \underline{a}_0 = \lambda^2, \qquad I_5 = \underline{a}_0 \cdot \underline{C}^2\, \underline{a}_0, \qquad I_6 = \underline{g}_0 \cdot \underline{C}\, \underline{g}_0, \\
I_7 &= \underline{g}_0 \cdot \underline{C}^2\, \underline{g}_0, \qquad I_8 = \frac{1}{2}\underline{a}_0 \cdot \underline{g}_0\ (\underline{a}_0 \cdot \underline{C}\, \underline{g}_0 + \underline{g}_0 \cdot \underline{C}\, \underline{a}_0), \qquad I_9 = \underline{a}_0 \cdot \underline{g}_0.
\end{aligned}
\tag{3.116}
$$

The dot product $\underline{a}_0 \cdot \underline{g}_0$ is a geometrical constant determining the cosine of the angle between the two fiber directions in the reference direction. Thus, the invariant I_9 is a constant throughout the deformation and does not need to be considered further. Invariants I_4 and I_6 represent the square of the stretch of the fibers in $\kappa(t)$ which were parallel to $\underline{a}_0$ and $\underline{g}_0$, respectively, in κ_0 . In the current configuration, the angle between the fibers will in general be different from ϕ_0 and related to ϕ through,

$$
\cos 2\phi = \underline{a} \cdot \underline{g} = (I_4 I_6)^{-1/2} \underline{a}_0 \cdot \underline{g}_0 = (I_4 I_6)^{-1/2} \cos 2\phi_0. \tag{3.117}
$$

The Cauchy stress tensor can be calculated directly from (3.99) using (3.115) and (3.116),

$$
T_{ij} = \frac{1}{J} F_{iR} F_{jS} \left(\frac{\partial W}{\partial C_{RS}} + \frac{\partial W}{\partial C_{SR}} \right) = I_3^{-1/2} F_{iR} F_{jS} \sum_{k=1}^{8} \frac{\partial W}{\partial I_k} \left(\frac{\partial I_k}{\partial C_{RS}} + \frac{\partial I_k}{\partial C_{SR}} \right). \tag{3.118}
$$

Therefore,

$$
\begin{aligned}
\underline{T} = 2\, I_3^{-1/2} \Big\{ &(I_2 \frac{\partial W}{\partial I_2} + I_3 \frac{\partial W}{\partial I_3}) \underline{I} + \frac{\partial W}{\partial I_1} \underline{b} - I_3 \frac{\partial W}{\partial I_2} \underline{b}^{-1} \\
&+ I_4 \frac{\partial W}{\partial I_4} \underline{a} \otimes \underline{a} + I_4 \frac{\partial W}{\partial I_5} (\underline{a} \otimes \underline{b}\, \underline{a} + \underline{b}\, \underline{a} \otimes \underline{a}) \\
&+ I_6 \frac{\partial W}{\partial I_6} \underline{g} \otimes \underline{g} + I_6 \frac{\partial W}{\partial I_7} (\underline{g} \otimes \underline{b}\, \underline{g} + \underline{b}\, \underline{g} \otimes \underline{g}) \\
&+ \frac{1}{2} I_4 \frac{\partial W}{\partial I_6}\, \underline{a} \cdot \underline{b}^{-1} \underline{g}\, (\underline{a} \otimes \underline{g} + \underline{g} \otimes \underline{a}) \Big\}.
\end{aligned}
\tag{3.119}
$$

Similarly,

$$
S_{AB} = \left(\frac{\partial W}{\partial C_{AB}} + \frac{\partial W}{\partial C_{BA}} \right) = \sum_{k=1}^{8} \frac{\partial W}{\partial I_k} \left(\frac{\partial I_k}{\partial C_{RS}} + \frac{\partial I_k}{\partial C_{SR}} \right). \tag{3.120}
$$

[6] Here, the subscript 0 is used with ϕ to emphasize that it is an angle in the reference configuration κ_0.

Using results from Appendix D, it is straightforward to show,

$$
\begin{aligned}
\underline{S} \; &= 2 \left\{ (\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2})\underline{I} \; - \; \frac{\partial W}{\partial I_2}\underline{C} \; + \; I_3 \frac{\partial W}{\partial I_3}\underline{C}^{1} \right. \\
&+ \frac{\partial W}{\partial I_4}\underline{a}_0 \otimes \underline{a}_0 \; + \; \frac{\partial W}{\partial I_5}\left(\underline{a}_0 \otimes \underline{C}\,\underline{a}_0 \; + \; \underline{C}\,\underline{a}_0 \otimes \underline{a}_0\right) \\
&+ \frac{\partial W}{\partial I_6}\underline{g}_0 \otimes \underline{g}_0 \; + \; \frac{\partial W}{\partial I_7}\left(\underline{g}_0 \otimes \underline{C}\,\underline{g}_0 \; + \; \underline{C}\,\underline{g}_0 \otimes \underline{g}_0\right) \\
&\left. + \frac{1}{2}\frac{\partial W}{\partial I_8}\underline{a}_0 \cdot \underline{g}_0 \left(\underline{a}_0 \otimes \underline{g}_0 \; + \; \underline{g}_0 \otimes \underline{a}_0\right) \right\}.
\end{aligned}
\tag{3.121}
$$

Special cases

The following special cases are straightforward to consider:

- **The families are orthogonal in κ_0**
  In this case, $\underline{a}_0 \cdot \underline{g}_0$ is zero and the material is orthotropic in reference configuration with respect to the planes perpendicular to te fiber directions and surface in which the fibers lie. The strain energy only depends on $I_1....I_7$. Hence, it is quite similar in form to the case of the one fiber family, with the addition of terms for $\underline{g}_0$,

$$
\begin{aligned}
\underline{T} \; &= 2\, I_3^{-1/2} \left\{ (I_2 \frac{\partial W}{\partial I_2} \; + \; I_3 \frac{\partial W}{\partial I_3})\underline{I} \; + \; \frac{\partial W}{\partial I_1}\underline{b} \; - \; I_3 \frac{\partial W}{\partial I_2}\underline{b}^{-1} \right. \\
&+ I_4 \frac{\partial W}{\partial I_4}\underline{a} \otimes \underline{a} \; + \; I_4 \frac{\partial W}{\partial I_5}\left(\underline{a} \otimes \underline{b}\,\underline{a} \; + \; \underline{b}\,\underline{a} \otimes \underline{a}\right) \; + \; I_6 \frac{\partial W}{\partial I_6}\underline{g} \otimes \underline{g} \\
&\left. + I_6 \frac{\partial W}{\partial I_7}\left(\underline{g} \otimes \underline{b}\,\underline{g} \; + \; \underline{b}\,\underline{g} \otimes \underline{g}\right) \right\}.
\end{aligned}
\tag{3.122}
$$

  As noted by [26], this last result is identical to that for an orthotropic material obtained by [**?**] and [10].

- **The families are orthogonal in κ_0 and the material is incompressible**
  For incompressible materials, $I_3 = 1$ and a constraint contribution proportional to $\underline{I}$ is added to the stress tensor for the compressible material. Without loss in generality, this constraint term can absorb all terms proportional to $\underline{I}$ so that the Cauchy stress tensor becomes, so that,

$$
\begin{aligned}
\underline{T} \; &= -p\underline{I} \; + \; 2 \left\{ \frac{\partial W}{\partial I_1}\underline{b} \; - \; \frac{\partial W}{\partial I_2}\underline{b}^{-1} \; + \; I_4 \frac{\partial W}{\partial I_4}\underline{a} \otimes \underline{a} \; + \; I_6 \frac{\partial W}{\partial I_6}\underline{g} \otimes \underline{g} \right. \\
&\left. + I_4 \frac{\partial W}{\partial I_5}\left(\underline{a} \otimes \underline{b}\,\underline{a} \; + \; \underline{b}\,\underline{a} \otimes \underline{a}\right) \; + \; I_6 \frac{\partial W}{\partial I_7}\left(\underline{g} \otimes \underline{b}\,\underline{g} \; + \; \underline{b}\,\underline{g} \otimes \underline{g}\right) \right\}.
\end{aligned}
\tag{3.123}
$$

- **The families are orthogonal in κ_0, *fibers are inextensible* and the material is incompressible**

If, additionally, the fibers are inextensible, then $I_4 = I_6 = 1$ and a constraint response of must be added to the stress tensor. This constraint response does no work,

$$\begin{aligned} \underline{T} &= 2\left\{\frac{\partial W}{\partial I_1}\underline{b} - \frac{\partial W}{\partial I_2}\underline{b}^{-1} + \frac{\partial W}{\partial I_5}\left(\underline{a}\otimes\underline{b}\,\underline{a} + \underline{b}\,\underline{a}\otimes\underline{a}\right)\right. \\ &\quad \left. + \frac{\partial W}{\partial I_7}\left(\underline{g}\otimes\underline{b}\,\underline{g} + \underline{b}\,\underline{g}\otimes\underline{g}\right)\right\} - p\underline{I} - q_a\,\underline{a}\otimes\underline{a} - q_g\,\underline{g}\otimes\underline{g}. \end{aligned} \tag{3.124}$$

- **The two families of fibers are mechanically equivalent**
  If the two families of fibers are mechanically equivalent (not necessarily orthogonal, not inextensible), then the material is *locally orthotropic* in the reference configuration with respect to the plane which bisects $\underline{a}_0$ and $\underline{g}_0$ and the planes containing $\underline{a}_0$ and $\underline{g}_0$. Since W will then be symmetric with respect to interchanges of $\underline{a}_0$ and $\underline{g}_0$, we are motivated to introduce the following invariants:

$$I_{10} = I_4 + I_6, \qquad I_{11} = I_4 I_6, \qquad I_{12} = I_5 + I_7, \qquad I_{13} = I_5 I_7, \tag{3.125}$$

It can be shown that I_{13} can be written in terms of the other invariants. Therefore, without loss in generality, we can write the strain energy as the following function,

$$\hat{W} = \tilde{W}(I_1, I_2, I_3, I_8, I_9, I_{10}, I_{11}, I_{12}). \tag{3.126}$$

The stress tensors can be calculated for (3.126) in a similar way.

Appendix A

Additional Considerations for SRBM

First consider, two material points which we can identify as P and Q. These points can also be identified by their respective positions, $\underline{X}$ and $\underline{Y}$, in the reference configuration. The positions of points P and Q in the current configuration, denoted by $\underline{x}$ and $\underline{y}$ respectively, are

$$\underline{x} \ = \ \underline{\chi}(\underline{X}, t), \qquad \underline{y} \ = \ \underline{\chi}(\underline{Y}, t). \tag{A.1}$$

The positions occupied by these materials points at time t in the superposed motion, (3.6), are then,

$$\underline{x}^{+} \ = \ \underline{\hat{\chi}}^{+}(\underline{x}, t). \qquad \underline{y}^{+} \ = \ \underline{\hat{\chi}}^{+}(\underline{y}, t). \tag{A.2}$$

By defintion, the distance between two arbitrary points in the superposed motion must be the same as the distance between those in the motion, (3.4),

$$(\underline{x}^{+} - \underline{y}^{+}) \cdot (\underline{x}^{+} - \underline{y}^{+}) \ = \ (\underline{x} - \underline{y}) \cdot (\underline{x} - \underline{y}), \tag{A.3}$$

namely,

$$(\underline{\hat{\chi}}^{+}(\underline{x}, t) - \underline{\hat{\chi}}^{+}(\underline{y}, t)) \cdot (\underline{\hat{\chi}}^{+}(\underline{x}, t) - \underline{\hat{\chi}}^{+}(\underline{y}, t)) \ = \ (\underline{x} - \underline{y}) \cdot (\underline{x} - \underline{y}). \tag{A.4}$$

Equation (A.4) can be written in indicial notation as,

$$(\hat{\chi}_i^{+}(\underline{x}, t) - \hat{\chi}_i^{+}(\underline{y}, t))(\hat{\chi}_i^{+}(\underline{x}, t) - \hat{\chi}_i^{+}(\underline{y}, t)) \ = \ (x_i - y_i)(x_i - y_i). \tag{A.5}$$

Recalling that $\underline{x}$ and $\underline{y}$ are independent, we can differentiate (A.5) with respect to x_j and then with respect to y_k to obtain,

$$\frac{\partial \hat{\chi}_i^{+}(\underline{x}, t)}{\partial x_j} \frac{\partial \hat{\chi}_i^{+}(\underline{y}, t)}{\partial y_k} \ = \ \delta_{jk}. \tag{A.6}$$

Equation (A.6) can be written in coordinate free notation as,

$$\left(\frac{\partial \underline{\hat{\chi}}^{+}(\underline{x}, t)}{\partial \underline{x}} \right)^{T} \cdot \frac{\partial \underline{\hat{\chi}}^{+}(\underline{y}, t)}{\partial \underline{y}} \ = \ \underline{I}. \tag{A.7}$$

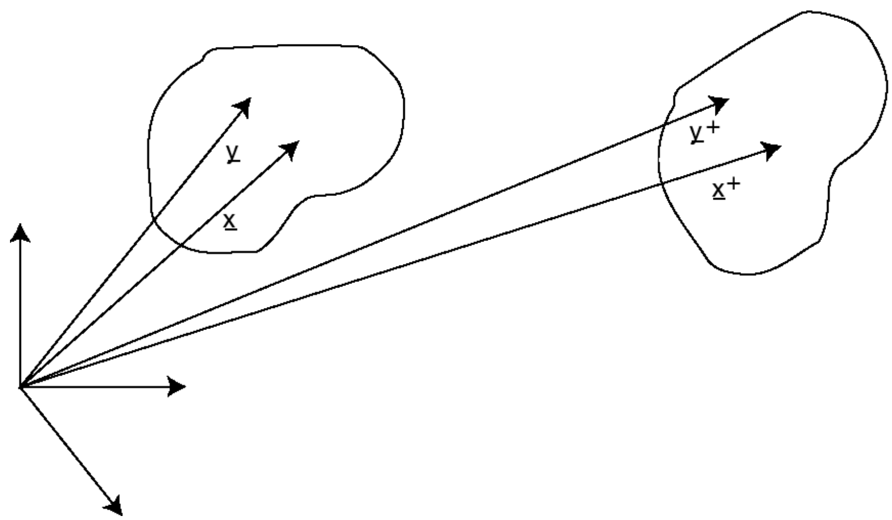

Figure A.1: Identification of points in the body in the current and superposed configurations.

Equivalently,

$$\left(\frac{\partial \hat{\underline{\chi}}^+(\underline{x},t)}{\partial \underline{x}}\right)^T = \left(\frac{\partial \hat{\underline{\chi}}^+(\underline{y},t)}{\partial \underline{y}}\right)^{-1}. \tag{A.8}$$

Equation (A.8) must hold for arbitrary choices of $\underline{x}$ and $\underline{y}$ in the body. Since the left side of equation (A.8) is indpendent of $\underline{y}$ and the right side is independent of $\underline{x}$, the second order tensors on the left and right hand side of (A.8) must be functions of t only, which we denote as the transpose of $\underline{Q}(t)$,

$$\underline{Q}(t) = \left(\frac{\partial \hat{\underline{\chi}}^+(\underline{x},t)}{\partial \underline{x}}\right), \text{ or } Q_{ij}(t) = \frac{\partial \hat{\chi}_i^+(\underline{x},t)}{\partial x_j}. \tag{A.9}$$

Since (A.9) must hold for *all* $\underline{x}$ in R,

$$\underline{Q}(t) = \left(\frac{\partial \hat{\underline{\chi}}^+(\underline{y},t)}{\partial \underline{x}}\right) \tag{A.10}$$

and therefore from (A.7)

$$\underline{Q}^T\underline{Q} = \underline{I}, \qquad \text{and} \qquad \det \underline{Q} = \pm 1, \tag{A.11}$$

or

$$Q_{li}Q_{lj} = \delta_{ij}. \tag{A.12}$$

We see from (A.11) that $\underline{Q}$ is an orthogonal tensor. Each superposed motion, (3.6), must include the particular case in which $\hat{\underline{\chi}}^+(\underline{x}, t) = \underline{x}$. It can be seen from (A.9), that for this case, $\underline{Q} = \underline{I}$ and $\det \underline{Q} = 1$. Since the motions under consideration are continuous, we must **always** have,

$$\det \underline{Q} = 1. \tag{A.13}$$

The equation (A.9), may be integrated with respect to $\underline{x}$ to yield,

$$\underline{x}^+ = \hat{\underline{\chi}}^+(\underline{x}, t) = \underline{c}(t) + \underline{Q}(t)\underline{x}, \quad \text{or} \quad x_i^+ = \hat{\chi}_i^+(\underline{x}, t) = c_i(t) + Q_{ik}(t)x_k, \tag{A.14}$$

where $\underline{c}$ is a vector function of time. Therefore, the most general superposed rigid body motion is described by (3.7). From (A.14), we also have

$$\underline{x} = \underline{Q}^T(\underline{x}^+ - \underline{c}), \qquad x_k = Q_{lk}(x_l^+ - c_l). \tag{A.15}$$

So far, we have shown that necessarily a superposed rigid body motion is represented by (3.7). We now show that it is also sufficient, namely, that all motions of the form (3.7) preserve distance between points.

$$\begin{aligned} |\underline{x}^+ - \underline{y}^+|^2 = (\underline{x}^+ - \underline{y}^+) \cdot (\underline{x}^+ - \underline{y}^+) &= \big(\underline{Q}(\underline{x} - \underline{y})\big) \cdot \big(\underline{Q}(\underline{x} - \underline{y})\big) \\ &= (\underline{x} - \underline{y}) \cdot \big(\underline{Q}^T\underline{Q}(\underline{x} - \underline{y})\big) \\ &= (\underline{x} - \underline{y}) \cdot (\underline{x} - \underline{y}) \\ &= |\underline{x} - \underline{y}|^2. \end{aligned} \tag{A.16}$$

Appendix B

Isotropic Tensors

Consider a rectangular Cartesion coordinate system with basis, $(\underline{e}_i)$, and a second Cartesion coordinate system which is rotated relative to the first, with basis $(\underline{e}'_i)$. The relationship between $(\underline{e}_i)$, and $(\underline{e}'_i)$ can be represented as

$$\underline{e}_i = A_{ij}\underline{e}'_j \tag{B.1}$$

where A_{ij} are components of a proper orthogonal tensor,

$$A_{ij}A_{kj} = \delta_{ik}, \qquad \det \underline{A} = 1. \tag{B.2}$$

Therefore, we also have

$$\underline{e}'_j = A_{ij}\underline{e}_i \tag{B.3}$$

Consider an arbitrary vector $\underline{v}$. We refer to the components of this vector with respect to basis $(\underline{e}_i)$, and $(\underline{e}'_i)$ as v_i and v'_i, respectively. Namely,

$$\underline{v} = v_i\underline{e}_i = v'_i\underline{e}'_i. \tag{B.4}$$

Using this notation,

$$\begin{aligned} \underline{v} = v_i\underline{e}_i &= v'_j\underline{e}'_j \\ &= v'_j A_{kj}\underline{e}_k \end{aligned} \tag{B.5}$$

Therefore,

$$\begin{aligned} v_l &= v'_j A_{kj}\underline{e}_k \cdot \underline{e}_l \\ &= A_{lj}v'_j \end{aligned} \tag{B.6}$$

Similarly, it can be shown that, the components $M_{ijk...t}$ of a tensor M will transform as

$$M_{ijk...t} = A_{ia}A_{jb}A_{kc}...A_{th}M'_{abc...h}. \tag{B.7}$$

A tensor $\underline{M}$ is called isotropic if its components retain the same value regardless of the coordinate axis rotation,

$$M_{ijk...t} = M'_{ijk...t} \tag{B.8}$$

for all proper orthogonal A_{ij}. Alternatively,

$$M_{ijk...t} = A_{ia}A_{jb}A_{kc}...A_{th}M'_{abc...h}. \tag{B.9}$$

Theorem 1.: All scalar invariants are isotropic tensors of order 0.

Theorem 2.: The only isotropic tensor $\underline{v}$ of order 1 is the zero tensor.
Proof: If $\underline{v}$ is an isotorpic tensor of order 1, then the components of $\underline{v}$ are unchanged under an proper orthogonal transformations:

$$v_i = A_{ij}v_j \tag{B.10}$$

for all proper orthogonal $\underline{A}$. Consider the case where the components of $\underline{A}$ are defined as

$$[\underline{A}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{B.11}$$

Upon consideration of equation (B.10), we see that this choice of $\underline{A}$ requires that

$$(v_1, v_2, v_3) = (-v_1, -v_2, v_3). \tag{B.12}$$

This implies that $v_1 = 0, v_2 = 0$. Similarly, if we now consider the case where the components of $\underline{A}$ are equal to

$$[\underline{A}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{B.13}$$

then necessarily $(v_1, v_2, v_3) = (v_1, -v_2, -v_3)$. so that v_3 must also be zero. Therefore, if $\underline{v}$ is an isotropic tensor of order 1, then necessarily $\underline{v}$ must be the zero vector. It is clear that this is also a sufficient condition.

Theorem 3.: Any isotropic tensor of order 2, can be written as a scaler multiple of the identity tensor. Namely, if $\underline{B}$ is a second order isotropic tensor, then the most general form of $\underline{B}$ can be written as

$$\underline{B} = \lambda \underline{I}, \qquad B_{ij} = \lambda\delta_{ij} \tag{B.14}$$

where λ is a scalar.
Theorem 4.: The components of the most general isotropic tensor $\underline{B}$ of order 3 can be written as

$$B_{pqr} = \lambda\,\epsilon_{ijk} \tag{B.15}$$

where λ is a scalar.
Proof: If $\underline{B}$ is an isotropic tensor of order 3, then the components of $\underline{B}$ are unchanged under an proper orthogonal transformations:

$$B_{ijk} = A_{ia}A_{jb}A_{kc}B_{abc} \tag{B.16}$$

for all proper orthogonal $\underline{A}$. Consider the following three choices for the components of $\underline{A}$,

$$[\underline{A}] = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad [\underline{A}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad [\underline{A}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \tag{B.17}$$

If we consider (B.10) for these three proper orthogonal choices for $\underline{A}$, we fined that There are only six non-zero components of $\underline{B}$, and they are related as

$$B_{123} = B_{231} = B_{312} = -B_{321} = B_{213} = B_{132}, \tag{B.18}$$

and therefore necessarily B_{ijk} must be of the form

$$B_{ijk} = \lambda\epsilon_{ijk} \tag{B.19}$$

where lambda is a scaler. The form (B.19) is also sufficient:

$$\begin{aligned} A_{ia}A_{jb}A_{kc}B_{abc} &= A_{ia}A_{jb}A_{kc}\lambda\epsilon_{abc} \\ &= \lambda\det(A_{ab})\epsilon_{ijk} \\ &= \lambda\epsilon_{ijk} \\ &= B_{ijk} \end{aligned} \tag{B.20}$$

Theorem 5.: The components of the most general isotropic tensor, $\underline{B}$ of order 4 can be written as

$$B_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}. \tag{B.21}$$

Exercise B.0.1 *Prove Theorem 2, making sure to show both necessary and sufficient conditions.*

Appendix C

Isotropic Tensor Functions

C.1 Scalar valued isotropic tensor functions

Definition: A scalar valued function of a second order tensor $\phi = \hat{\phi}(\underline{B})$ is said to be an *isotropic tensor function* if

$$\phi(B_{ij}) \;=\; \phi(B'_{ij}) \;=\; \phi(A_{ik}B_{kl}A_{jl}) \tag{C.1}$$

for all proper orthogonal $\underline{A}$. Such a scalar valued function is also called simply an *invariant*.

Definition: The following isotropic scalar functions $\mathrm{I}_{\underline{B}}$, $\mathrm{II}_{\underline{B}}$ $\mathrm{III}_{\underline{B}}$ are called the *Principal Invariants* of a second order tensor $\underline{B}$,

$$\begin{aligned}
\mathrm{I}_{\underline{B}} &= \mathrm{tr}\underline{B} \;=\; \mathrm{B_{ii}} \\
\mathrm{II}_{\underline{B}} &= \frac{1}{2}\left(\mathrm{tr}(\underline{B})^2 - \mathrm{tr}(\underline{B}\cdot\underline{B})\right) \;=\; \frac{1}{2}\left(B_{ii}B_{jj} - B_{ij}B_{ji}\right) \\
\mathrm{III}_{\underline{B}} &= \det\underline{B}
\end{aligned} \tag{C.2}$$

The set of invariants (C.2) arise frequently as the coefficients of the characteristic equation for a second order tensor. Another set of invariants are,

$$\begin{aligned}
\bar{\mathrm{I}}_{\underline{B}} &= \mathrm{tr}\underline{B} \;=\; \mathrm{B_{ii}} \\
\bar{\mathrm{II}}_{\underline{B}} &= \mathrm{tr}\underline{B}^2 \;=\; \mathrm{B_{ij}B_{ji}} \\
\bar{\mathrm{III}}_{\underline{B}} &= \mathrm{tr}\underline{B}^3 \;=\; \mathrm{B_{ij}B_{jk}B_{ki}}
\end{aligned} \tag{C.3}$$

Representation Theorem for Invariant Scalar Valued Functions of a Symmetric Tensor

A scalar valued function of a symmetric tensor $\phi(\underline{B})$ is an invariant if and only if it can be written as a function of the principal invariants of the symmetric tensor $\underline{B}$.

$$\phi(\underline{B}) \;=\; \hat{\phi}(\mathrm{I}_{\underline{B}}, \mathrm{II}_{\underline{B}}, \mathrm{III}_{\underline{B}}) \tag{C.4}$$

C.2 Symmetric Isotropic Tensor Functions

Definition: A second order tensor valued function of a second order tensor $\underline{F} = \hat{\underline{F}}(\underline{B}_1, \underline{B}_2, ...\underline{B}_N)$ is said to be an *isotropic tensor function* if

$$\underline{A} \cdot (\underline{B}_1, \underline{B}_2, ...\underline{B}_N) \cdot \underline{A}^T \;=\; \hat{\underline{F}}(\underline{A} \cdot \underline{B}_1 \cdot \underline{A}^T, \underline{A} \cdot \underline{B}_2 \cdot \underline{A}^T, ...\underline{A} \cdot \underline{B}_N \cdot \underline{A}^T) \tag{C.5}$$

for all proper orthogonal $\underline{A}$ and all $\underline{B}_i$ in the domain of definition of the function $\underline{F}$.

Representation Theorem for Symmetric Isotropic Tensor Functions

A tensor function $\underline{B} = \hat{\underline{B}}(\underline{A})$ for which both $\underline{A}$ and $\underline{B}$ are symmetric second order tensors, is an isotropic function if and only if it has a representation of the form,

$$\underline{B} \;=\; \hat{\underline{B}}(\underline{A}) \;=\; \phi_o \underline{I} \;+\; \phi_1 \underline{A} \;+\; \phi_2 \underline{A}^2 \tag{C.6}$$

where ϕ_1, ϕ_2, ϕ_3 *are functions of the principal invariants of* $\underline{A}$.

This theorem can be generalized for tensors which are not of second order (see, for example, Truesdell, Non-Linear Field Theories of Mechanics, Handbuch Der Physik, Volume III/3, 1965, p 32), [**?**].

Appendix D

Some Derivations for Isotropic Materials

We now derive the following useful identities for the discussion of isotropic materials. Consider the symmetric second order tensor $\underline{C}$ with principle invariants I_1, I_2, I_3. In this appendix, we will prove the following results,

$$\begin{aligned} \frac{\partial I_1}{\partial \underline{C}} &= \underline{I} & & & \frac{\partial I_1}{\partial C_{AB}} &= \delta_{AB} \\ \frac{\partial I_2}{\partial \underline{C}} &= I_1 \underline{I} - \underline{C} & &\text{or} & \frac{\partial I_2}{\partial C_{AB}} &= I_1 \delta_{AB} - C_{AB} \\ \frac{\partial I_3}{\partial \underline{C}} &= I_3 \underline{C}^{-1} & & & \frac{\partial I_3}{\partial C_{AB}} &= I_3 C^{-1}_{AB} \end{aligned} \tag{D.1}$$

The first result is easily proven through,

$$\frac{\partial I_1}{\partial C_{AB}} = \frac{\partial C_{DD}}{\partial C_{AB}} = \delta_{DA}\delta_{DB} = \delta_{AB}. \tag{D.2}$$

Furthermore,

$$\begin{aligned} \frac{\partial I_2}{\partial C_{AB}} &= \frac{1}{2}\frac{\partial}{\partial C_{AB}}(I_1^2 - C_{FG}C_{FG}) \\ &= I_1 \frac{\partial I_1}{\partial C_{AB}} - \frac{1}{2}\left(\frac{\partial C_{FG}}{\partial C_{AB}} + C_{FG}\frac{\partial C_{FG}}{\partial C_{AB}}\right) \\ &= I_1\,\delta_{AB} - C_{AB} \end{aligned} \tag{D.3}$$

Finally,

$$\frac{\partial I_3}{\partial \underline{C}} = \frac{\partial \det \underline{C}}{\partial \underline{C}} = I_3 \underline{C}^{-1} \tag{D.4}$$

where we have used the result $\partial \det \underline{A} / \partial \underline{A} = \det \underline{A}\, \underline{A}^{-T}$, for all non-singular second order tensors $\underline{A}$.

D.0.1 Results for the invariants for second order tensors $\underline{C}$ and $\underline{a}_0 \otimes \underline{a}_0$

It is also useful to note the following results for invariants,

$$I_4 \ = \ \underline{a}_0 \cdot (\underline{C}\underline{a}_0), \qquad I_5 \ = \ \underline{a}_0 \cdot (\underline{C}^2 \underline{a}_0), \tag{D.5}$$

that will be used in deriving the form of the stress tensor for fiber reenforced materials,

$$\begin{aligned} \frac{\partial I_4}{\partial \underline{C}} &= \underline{a}_0 \otimes \underline{a}_0 & \qquad & \frac{\partial I_4}{\partial C_{AB}} &= a_{0A} a_{0B}, \\ \frac{\partial I_5}{\partial \underline{C}} &= \underline{a}_0 \otimes \underline{C}\underline{a}_0 \ + \ \underline{C}\underline{a}_0 \otimes \underline{a}_0 & \quad \text{or} \quad & \frac{\partial I_5}{\partial C_{AB}} &= a_{0A} C_{BC} \underline{a}_{0C} \ + \ C_{AC} \underline{a}_{0C} a_{0B}. \end{aligned} \tag{D.6}$$

D.0.2 Results for the invariants for second order tensors $\underline{C}$, $\underline{a}_0 \otimes \underline{a}_0$, $\underline{g}_0 \otimes \underline{g}_0$

It is also useful to note the following results for additional invariants which arise for the case of two fiber families,

$$I_6 \ = \ \underline{g}_0 \cdot (\underline{C}\underline{g}_0), \quad I_7 \ = \ \underline{g}_0 \cdot (\underline{C}^2 \underline{g}_0), \quad I_8 \ = \ \frac{1}{2} (\underline{a}_0 \cdot \underline{g}_0)(\underline{a}_0 \cdot ((\underline{C}\underline{g}_0) + \underline{g}_0 \cdot (\underline{C}\underline{a}_0)), \tag{D.7}$$

It is useful to note that,

$$\begin{aligned} \frac{\partial I_6}{\partial \underline{C}} &= \underline{g}_0 \otimes \underline{g}_0, \\ \frac{\partial I_7}{\partial \underline{C}} &= \underline{g}_0 \otimes \underline{C}\underline{g}_0 \ + \ \underline{C}\underline{g}_0 \otimes \underline{g}_0, \\ \frac{\partial I_8}{\partial \underline{C}} &= (\underline{a}_0 \cdot \underline{g}_0), (\underline{a}_0 \otimes \underline{g}_0 + \underline{g}_0 \otimes \underline{a}_0). \end{aligned} \tag{D.8}$$

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