The Reduced Basis Method for an Elastic Buckling Problem

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In this work, we apply the Reduced Basis (RB) Method to the field of nonlinear elasticity. In this first stage of research, we analyze a buckling problem for a compressed 2D column: Here, the trivial linear solution is computed for an arbitrary load; the critical load, marking the transition to nonlinearity, is then identified through an eigenvalue problem. The linear problem satisfies the Lax-Milgram conditions, allowing the implementation of both a Successive Constraint Method for an inexpensive lower bound of the coercivity constant and of a rigorous and efficient a posteriori error estimator for the RB approximation. Even though only a non-rigorous estimator is available for the buckling problem, the actual RB approximation of the output is more than satisfactory, and the gain in computational efficiency significant.

1 Introduction

We develop Reduced Basis (RB) approximation and error estimation methods for parametrized partial differential equations in an instability problem in the context of nonlinear elasticity in solid mechanics. Numerical experiments are performed on a model problem involving a compressed 2D column; here the output of interest is the critical load causing the buckling of the structure, i.e. the smallest generalized eigenvalue of the system. A physical extension and possible application of the analyzed compressed beam is a cantilevered parallel-chord truss, where an applied load would result in the buckling of several members of the truss.

In Section 2 we derive the set of equations leading to a finite element (FE) *truth* solution. Then, this geometrically parametrized problem is the object of RB approximation in Section 3.

2 Problem Statement and FE Approximation

Following [1], the behavior of the considered 2D column follows the equilibrium and momentum equations[1]:

$$((\delta_{ik} + u_{i,k})S_{kj})_{,j} + b_i = 0 \ , \quad S_{ij} = (S_{ij})^t \ , \text{ in } \Omega \ ; \tag{1}$$

where u indicates the displacement, b the (neglected) volumetric forces and S the second Kirchoff-Piola stress tensor. One of the column extremities is clamped, while a compressive traction is applied on the opposite side, whence the boundary conditions: $u_i = \bar{\lambda} u_i^S = 0$ on $\delta\Omega^D$; $\quad T_i := (S_{ij} + u_{i,k}S_{kj})n_j = (\bar{\lambda}T_i^S$, on $\delta\Omega^T$; 0, on $\delta\Omega \setminus (\delta\Omega^T \cup \delta\Omega^D))$.

We notice that the traction T is proportional to a parameter $\bar{\lambda} \in \mathbb{R} > 0$ and to a fixed quantity T^S: the same applies to the Dirichlet conditions (which in our case are set to zero). In correspondence to an infinitesimal increment of the loading with respect to a *fundamental* solution $u^1(\bar{\lambda})$, the new solution is assumed to be non-unique. In particular, we define incremental and difference quantities: $0 \neq \dot{u}^a - \dot{u}^b =: \tilde{u} =: \xi$. Equations (1) and their boundary conditions being valid for both incremental states a and b, we obtain by difference the system:

$$\begin{cases} (\tilde{S}_{ij} + u^1_{i,k}(\bar{\lambda})\tilde{S}_{kj} + S^1_{kj}\xi_{i,k})_{,j} = 0 \ , \text{ on } \Omega \ ; \\ \xi_i = 0 \ , \text{ on } \delta\Omega^D \ ; \quad \tilde{T}_i = (\tilde{S}_{ij} + u^1_{i,k}(\bar{\lambda})\tilde{S}_{kj} + S^1_{kj}\xi_{i,k})n_j = 0 \ , \text{ on } \delta\Omega^T \ . \end{cases} \tag{2}$$

This system must be closed by defining the stress tensor through a constitutive equation, in our case a St. Venant relation, $S_{ij} = \mathcal{C}_{ijkl}E_{kl}$ - the Green-Lagrange tensor being: $E_{ij} := \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$, and the St. Venant tensor: $\mathcal{C}_{ijkl} := \Lambda\delta_{ij}\delta_{kl} + M(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$[2]. The *fundamental* solution is here imposed as the solution of a linearized elastic problem: We can accept such approximation, the corresponding load $\lambda^{lin} = 10^{-4}$ being considerably small. Exploiting the linearity, we replace $u^1(\bar{\lambda}) = \lambda u^1(\lambda^{lin}) =: \lambda u^1$ in system (2), which therefore represents a generalized eigenvalue problem. The first eingenvalue in the eigenpair (λ, ξ) corresponds to the critical buckling load and is the object of our approximation.

The considered column is a parametrized rectangle $\Omega(\mu) = [0, L] \times [0, \mu]$, $L = 1$, $\mu \in [0.03125, 0.2]$. First, a reference parameter $\bar{\mu}$ is fixed as the 25th value over 200 log-spaced parameters of the collection $\mathcal{D}$, then the desired eigenvalue is computed through a FE Galerkin P1 approximation[3] of (high) dimension $\mathcal{N}$.

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[1] Sum over repeated indices is assumed. We also indicate $u_{i,j} := \frac{\partial u_i}{\partial x_j}$, $\delta_{ij} = (1, \ i = j; \ 0, \ i \neq j)$ and n_i as the outer normal unit vector; $i, j = 1, 2$.

[2] The Lamé parameters Λ and M are functions of the Poisson's ratio $\nu = 0.25$ and elastic modulus $E = 1$.

[3] A reasonably fine mesh of size 0.02 outputs the value $\lambda(\bar{\mu}) = 0.319068$, whose correctness can be inferred by comparison with an analytical value from [2], $\lambda_{Sh}(\bar{\mu}) = \pi^2 EI/4L^2 = 0.247263$, I being the moment of area of the rectangular cross section.

3 The RB Approximation

We first apply the RB method to the weak formulation of the linearized elastic problem: Find $u \in V \subset (H^1(\Omega))^2 : \ \forall v \in V, \ a(u, v; \mu) = f(v; \mu)$, where $a(u, v; \mu) = \int_{\Omega(\mu)} u_{i,j} \mathcal{C}_{ijkl} v_{k,l}; \ f(v; \mu) = -\frac{10^{-4}}{\mu} \int_{\partial\Omega^T(\mu)} v_1$. First, the linear and bilinear forms are decomposed into linear combinations of parameter-dependent coefficients and parameter-independent forms: $a(u, v; \mu) = \sum_{q=1}^{Q_A} \Theta_a^q(\mu) a^q(u, v); \ f(v; \mu) = \sum_{q=1}^{Q_F} \Theta_f^q(\mu) f^q(v)$. This is crucial in order to decompose the computation into *offline* and *online* (real-time) stages whose complexity is, respectively, $\mathcal{N}$-dependent and $\mathcal{N}$-independent.

An RB space is built *offline* as $W_N^u = span\{u(\mu_i); \mu_i = 1, \dots, N\}$; here the parameters μ_i are selected using a greedy algorithm, [3]. As the *online* result of an N-dimensional system, the RB approximated solution for a new parameter μ is: $u_N(\mu) := \sum_{j=1}^{N} u_{Nj}(\mu)\zeta_j$, where each $\zeta_j \in W_N^u$ and the coefficients $u_{Nj}(\mu)$ represent the solution of the reduced system.

The greedy sampling procedure is based on the error estimator $\Delta_N(\mu) = \frac{\|r_N(\cdot;\mu)\|_{V'}}{\alpha_{LB}(\mu)} \geq \|u(\mu) - u_N(\mu)\|_V$, [3], which exploits the dual norm of the residual $r_N(v; \mu) := a(u_N(\mu), v) - f(v; \mu)$. The lower bound for the coercivity constant $\alpha_{LB}(\mu)$ is computed by means of the Successive Constraint Method [4], which here requires 196/200 evaluations of $\alpha(\mu)$, although with a significant decrease of computational effort in the *online* phase. We observe the decrease of error and error estimator in Figure 1 and 2: The upper bound reveals that a precision of 1% w.r.t. FE can be achieved with only two basis functions.

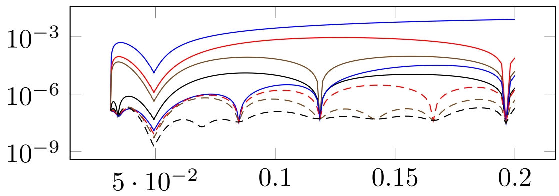

Fig. 1: $\Delta_N(\mu)$, normalized w.r.t. $\|u_N(\mu)\|_V$; $N = 1, \dots, 8$.

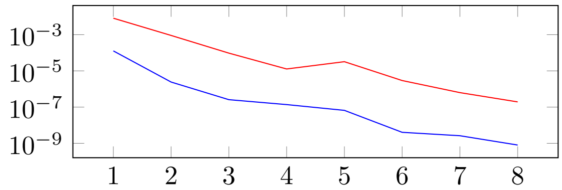

Fig. 2: $\max_{\mu\in\mathcal{D}} \frac{\Delta_N(\mu)}{\|u_N(\mu)\|_V}$ and corresp. $\frac{\|u(\mu)-u_N(\mu)\|_V}{\|u_N(\mu)\|_V}$.

Starting from the reduced fundamental solution $u_N^1(\mu)$ obtained from the linearized problem, we can write the weak formulation of (2): Find $(\lambda, \xi) \in \mathbb{R} \times V$ such that $\forall v \in V, \ a(\xi, v; \mu) = \lambda(\mu) b(u_N^1(\mu), \xi, v; \mu)$. Here, for the affine-decomposable trilinear form $b(u, \xi, v; \mu) = \int_{\Omega(\mu)} \mathcal{C}_{ijkl} u_{i,j} \xi_{m,k} v_{m,l}$, the non-symmetrical terms are neglected, as they do not influence the smallest eigenvalue of interest. By means of the greedy parameters selected for the linearized problem, we build in the *offline* phase a new RB space $W_N^\xi = span\{\xi(\mu_i), i = 1, \dots, N\}$. In the *online* phase, for every new parameter, the output of interest is the minimum of the following *reduced* Rayleigh quotient: $\lambda_N(\mu) = \min_{v_N(\mu)\in W_N^\xi} \frac{a(v_N, v_N; \mu)}{b(u_N^1(\mu), v_N, v_N; \mu)}$. As expected, a decreasing pattern towards the FE eigenvalue is observed corresponding to an increasing number of basis functions. An upper bound $\delta(\lambda)$ [5] for the precision of the SLEPC solver justifies the slight discrepancies, in red in Table 1.

Table 1: Convergence of the reduced eigenvalue to FE, for three parameter samples $\mu^- = 0.0391 \leq \bar{\mu} = 0.04936 \leq \mu^+ = 0.1255$.

	$\lambda_N(\mu^-)$	$\delta(\lambda)$	$\lambda_N(\bar{\mu})$	$\delta(\lambda)$	$\lambda_N(\mu^+)$	$\delta(\lambda)$
N=7	0.174208974091293	5.3e-08	0.319067591915845	5.8e-09	4.45895577972351	2.2e-08
N=8	0.174208973929127	5.4e-08	0.319067590503729	4.5-09	4.45895577971728	1.9e-08
FE	0.174209012605825	\	0.319067602303527	\	4.45895577540437	\

Following [6], an asymptotic a posteriori error bound - although non-rigorous - can be computed to verify the quality of the approximation. Fixing M:=2N, the estimate is defined as $\Delta_{NM}(\mu) := \frac{1}{\tau}|\lambda_M(\mu) - \lambda_N(\mu)|, \ \tau \in (0, 1)$. Were the asymptotic convergence assumption met, for N big enough, it would hold: $\eta_{NM}(\mu) := \frac{\Delta_{NM}(\mu)}{e_N(\mu)} := \frac{\Delta_{NM}(\mu)}{|\lambda(\mu) - \lambda_N(\mu)|} \in (1, \frac{1}{\tau})$, see Table 2.

Table 2: Behavior of the asymptotic error bound (efficiency, error and bound) over the parameter domain, $\frac{1}{\tau} = 2$.

N	M	$\max_{\mu\in\mathcal{D}} \eta_{NM}(\mu)$	$e_N(\mu)$	$\Delta_{NM}(\mu)$
3	6	2.000422	2.58e-04	5.16e-04
4	8	2.227196	3.27e-07	7.28e-07

N	M	$\min_{\mu\in\mathcal{D}} \eta_{NM}(\mu)$	$e_N(\mu)$	$\Delta_{NM}(\mu)$
3	6	1.999373	2.47e-04	4.94e-04
4	8	1.877748	1.03e-06	1.94e-06

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