A Hybridized DG / Mixed Method For Nonlinear Convection-Diffusion Problems

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Outline

1. Introduction
2. BDM Mixed Method for Diffusion
3. Hybridized BDM Mixed Method for Diffusion
4. Hybridized DG-BDM (HDG-BDM) for Advection-Diffusion
5. Hybridized DG (HDG) for Advection-Diffusion
6. Numerical Results
**Background**

- HDG-BDM method for Advection-Diffusion equations.

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- Non-Linear case: Proposed by J. Schütz and G. May (Promising results for N-S equations \(^2\))


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- Reduces to DG for pure advection, to BDM mixed for pure diffusion

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- It can be even mixed with the HDG scheme due to hybridization.

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Consider Laplace equation

\[-\nabla \cdot \nabla u = S \quad \text{in} \quad \Omega\]
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\[
V_h := \{ \varphi \in L^2(\Omega) : \varphi|_{\Omega_k} \in P^{m-1}(\Omega_k) \} \\
\tilde{H}_h := \{ \tau \in H(div, \Omega) : \tau|_{\Omega_k} \in P^m(\Omega_k) \times P^m(\Omega_k) \}
\]
BDM Mixed Method for Diffusion

Consider Laplace equation

\[-\nabla \cdot \nabla u = S \quad \text{in} \quad \Omega \]
\[u = g \quad \text{in} \quad \partial\Omega\]

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BDM Mixed method

\[\int_{\Omega} \sigma_h \cdot \tau + \int_{\Omega} (\nabla \cdot \tau) u_h - \int_{\partial\Omega} (\tau \cdot n) g = 0 \quad \forall \tau \in \tilde{H}_h\]
\[-\int_{\Omega} \nabla \cdot \sigma_h \varphi = \int_{\Omega} S \varphi \quad \forall \varphi \in V_h\]
The solution spaces: \( u_h \in V_h, \sigma_h \in H_h, \lambda_h \in M_h \)

\[
V_h := \{ \varphi \in L^2(\Omega) : \varphi|_{\Omega_k} \in P^{m-1}(\Omega_k) \}
\]

\[
H_h := \{ \tau \in L^2(\Omega) \times L^2(\Omega) : \tau|_{\Omega_k} \in P^m(\Omega_k) \times P^m(\Omega_k) \}
\]

\[
M_h := \{ \mu \in L^2(\Gamma) : \mu|_{\Gamma_k} \in P^m(\Gamma_k) \}
\]
Hybridizing...

The solution spaces: \( u_h \in V_h, \sigma_h \in H_h, \lambda_h \in M_h \)

\[
\begin{align*}
V_h &:= \{ \varphi \in L^2(\Omega) : \varphi|_{\Omega_k} \in P^{m-1}(\Omega_k) \} \\
H_h &:= \{ \tau \in L^2(\Omega) \times L^2(\Omega) : \tau|_{\Omega_k} \in P^m(\Omega_k) \times P^m(\Omega_k) \} \\
M_h &:= \{ \mu \in L^2(\Gamma) : \mu|_{\Gamma_k} \in P^m(\Gamma_k) \}
\end{align*}
\]

Hyb. BDM mixed method

\[
\begin{align*}
\sum_k \int_{\Omega_k} \sigma_h \cdot \tau + \int_{\Omega_k} (\nabla \cdot \tau) u_h - \int_{\partial \Omega_k} (\tau \cdot n) \lambda_h &= 0 & \forall \tau \in H_h \\
- \sum_k \int_{\Omega_k} (\nabla \cdot \sigma_h) \varphi &= \sum_k \int_{\Omega_k} S \varphi & \forall \varphi \in V_h \\
\sum_k \int_{\partial \Omega_k} -(\sigma_h \cdot n) \mu &= 0 & \forall \mu \in M_h
\end{align*}
\]
Advection-Diffusion equation

$$\nabla \cdot f(u) - \epsilon \nabla \cdot \nabla u = S$$
Adding DG for Advection

Advection-Diffusion equation

\[ \nabla \cdot f(u) - \epsilon \nabla \cdot \nabla u = S \]

HDG-BDM method

\[
\sum_k \int_{\Omega_k} \epsilon^{-1} \sigma_h \cdot \tau + \int_{\Omega_k} (\nabla \cdot \tau) u_h - \int_{\partial \Omega_k} (\tau \cdot n) \lambda_h = 0
\]

\[
\sum_k \int_{\Omega_k} -f(u_h) \cdot \nabla \varphi + \int_{\Gamma_k} \varphi (f(\lambda_h) \cdot n - \alpha(\lambda_h - u_h)) - \int_{\Omega_k} (\nabla \cdot \sigma_h) \varphi
\]

\[
= \sum_k \int_{\Omega_k} S \varphi
\]

\[
\sum_k \int_{\partial \Omega_k} (-\sigma_h \cdot n + f(\lambda_h) \cdot n - \alpha(\lambda_h - u_h)) \mu = 0
\]
Hybridized DG

Proposed by Nguyen et. al \(^5\)

The solution spaces: \(u_h \in \tilde{V}_h, \sigma_h \in H_h, \lambda_h \in M_h\)

\[
\tilde{V}_h := \{ \varphi \in L^2(\Omega) : \varphi|_{\Omega_k} \in P^m(\Omega_k) \}
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HDG method

\[
\sum_k \int_{\Omega_k} \epsilon^{-1} \sigma_h \cdot \tau + \int_{\Omega_k} (\nabla \cdot \tau) u_h - \int_{\partial \Omega_k} (\tau \cdot n) \lambda_h = 0
\]

\[
\sum_k \int_{\Omega_k} -f(u_h) \cdot \nabla \varphi + \int_{\Gamma_k} \varphi (f(\lambda_h) \cdot n - \beta(\lambda_h - u_h)) - \int_{\Omega_k} (\nabla \cdot \sigma_h) \varphi = \sum_k \int_{\Omega_k} S\varphi
\]

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\sum_k \int_{\partial \Omega_k} (-\sigma_h \cdot n + f(\lambda_h) \cdot n - \beta(\lambda_h - u_h)) \mu = 0
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<td>$u_h</td>
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<tr>
<td>$-\sigma_h + \hat{f}_h = -\sigma_h + f(\lambda_h) - \alpha(\lambda_h - u_h)n$</td>
<td>$-\sigma_h + \hat{f}_h = -\sigma_h + f(\lambda_h) - \beta(\lambda_h - u_h)n$</td>
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Comparison

**HDG-BDM**

\[ u_h|_{\Omega_k} \in P^{m-1} \]
\[ -\sigma_h + \hat{f}_h = -\sigma_h + f(\lambda_h) - \alpha(\lambda_h - u_h)n \]

**HDG**

\[ u_h|_{\Omega_k} \in P^m \]
\[ -\hat{\sigma}_h + \hat{f}_h = -\sigma_h + f(\lambda_h) - \beta(\lambda_h - u_h)n \]

**Common method**

\[
\sum_k \int_{\Omega_k} \epsilon^{-1} \sigma_h \cdot \tau + \int_{\Omega_k} (\nabla \cdot \tau)u_h - \int_{\partial \Omega_k} (\tau \cdot n)\lambda_h = 0
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\[
\sum_k \int_{\Omega_k} -f(u_h) \cdot \nabla \varphi + \int_{\Gamma_k} \varphi (f(\lambda_h) \cdot n - (\alpha|\beta)(\lambda_h - u_h)) - \int_{\Omega_k} (\nabla \cdot \sigma_h)\varphi
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\[
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Post processing of the solution \(^6\)

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Same convergence of post-processed solution under optimal conditions.

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Same convergence of post-processed solution under optimal conditions.

For HDG-BDM and HDG-RT\(^8\), this optimal condition is when diffusion dominates and one can put \(\alpha = 0\)

---

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Test case 1: Boundary Layer

Two dimensional viscous Burgers equation

\[
\frac{1}{2} \nabla \cdot (u^2, u^2) - \epsilon \nabla \cdot \nabla u = S \quad \text{in } \Omega
\]

\[
u = 0 \quad \text{in } \partial \Omega
\]

Solution:

\[
u(x, y) = \left( x + \frac{e^{c_1 x/\epsilon} - 1}{1 - e^{c_1/\epsilon}} \right) \cdot \left( y + \frac{e^{c_1 y/\epsilon} - 1}{1 - e^{c_1/\epsilon}} \right)
\]
Test case 1 : Boundary Layer

Figure: Contours of $u, m = 2 (u \in P^1), \epsilon = 0.1$, HDG-BDM scheme
Test case 1: Boundary Layer

Figure: Contours of $u^*$, $m = 2$ ($u \in P^1$), $\epsilon = 0.1$, HDG-BDM scheme
Test case 1: Boundary Layer

Figure: Contours of $u^*$, $m = 2 (u \in P^1)$, $\epsilon = 0.1$, $\alpha = 0$, HDG-BDM scheme
Test case 1: Boundary Layer

Figure: Convergence, $m = 3 \ (u \in P^2), \ \epsilon = 0.1$, HDG-BDM scheme
Test case 2: Linear Boundary Layer

Mixing HDG and HDG-BDM methods:

Condition: If Peclet number, $Pe = \frac{|c|h}{\epsilon} < 5$, then use HDG-BDM
Test case 2: Linear Boundary Layer

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Contours of \( u^* \), \( m = 2 \), \( \epsilon = 0.01 \)

Red: HDG, Blue: HDG-BDM
Test case 2: Linear Boundary Layer

Convergence of $u^*$, $m = 2$

Reduction of dofs of $u$
Test case 3

Advection Diffusion equation:

\[ \nabla \cdot u - \nabla \cdot (\epsilon(x) \nabla u) = S \quad \text{in} \quad \Omega \]
\[ u = g \quad \text{in} \quad \Gamma_D \]
Test case 3

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Diffusion Coefficient:

\[ \epsilon = \begin{cases} 
0.001, & x \leq 0.9 \\
1, & x \geq 1.1 \\
\text{smooth fn.}, & 0.9 < x < 1.1 
\end{cases} \]
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Solution:

\[ u(x, y) = (1 - \epsilon(x)) \sin(x - y) + \epsilon(x) \sin(2\pi x) \sin(2\pi y) \]
Test case 3

Condition: $x > 1.2$, use HDG-BDM

**Figure:** Red: HDG, Blue: HDG-BDM

**Figure:** Contours of $u^*$, $m = 2$
Test case 3

Condition: $x > 1.2$, use HDG-BDM

Figure: Convergence of $u^*$, $m = 2$
Test case 3

Condition: $Pe < 5$, use HDG-BDM

Figure: Red: HDG, Blue: HDG-BDM

Figure: Contours of $u^*$, $m = 2$
Test case 3

Condition: $Pe < 5$, use HDG-BDM

Figure: Convergence of $u^*$, $m = 2$
Conclusions

Present work:
- HDG-BDM method and its connection with HDG scheme

Future work:
- A robust sensor to determine the region for using HDG-BDM scheme
- Shock capturing
- Extending to Navier-Stokes equations
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Financial support from the Deutsche Forschungsgemeinschaft (German Research Association) through grant GSC 111 is gratefully acknowledged.
Cell-wise discretization of the Neumann problem:

\[
\epsilon(\nabla u_h^*, \nabla \phi) = (\sigma_h, \nabla \phi) \quad \forall \phi \in P_0^q(\Omega_k)
\]

\[
(u_h, 1) = (u_h^*, 1)
\]

where

\[
P_0^q(\Omega_k) := \{ \phi \in P^q(\Omega_k), (\phi, 1) = 0 \}
\]

with \( q = m + 1 \) for HDG and HDG-BDM \((\alpha = 0)\) and \( q = m \) for HDG-BDM \((\alpha \neq 0)\).
\[ \epsilon = e^{(-9 + 10x)^{-2}} (e^{(-9 + 10x)^{-2}} + e^{(-11 + 10x)^{-2}}) - 1 \]
Test case 1: Boundary Layer

**Figure:** Convergence, \( m = 3 \) (\( u \in P^3 \)), \( \epsilon = 0.1 \), HDG scheme
Test case 1: Boundary Layer

Figure: Convergence, \( m = 3, \epsilon = 0.1 \), HDG \( (u \in P^3) \) and HDG-BDM \( (u \in P^2) \) schemes