Adjoint-Based Mesh Adaptation for a Class of High-Order Hybridized Finite-Element Schemes for Convection-Diffusion Problems

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Accuracy and efficiency are very important aspects in modern computational methods.
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- Tackle this problem from two sides:
  - Hybridization
  - Output-based adaptation
Idea of Hybridization

- Formulate the global system only on the element interfaces
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- Obtain solution within the elements via so-called local solvers
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Figure: Globally coupled degrees of freedom

(a) Standard DG method                          (b) Hybridized method

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Idea of Hybridization

- Formulate the global system only on the element interfaces
- Obtain solution within the elements via so-called local solvers

(a) Standard DG method  
(b) Hybridized method

Figure: Globally coupled degrees of freedom
Problem Size

![Graph showing the relationship between Globally Coupled DOFs and Polynomial Degree for DG and HDG methods. The graph plots Globally Coupled DOFs on the y-axis and Polynomial Degree on the x-axis. Red and green lines represent DG and HDG methods, respectively.]
The convection-diffusion equation

\[ \nabla \cdot (f_c(w) - f_v(w, \nabla w)) = s(w, \nabla w) \]
Setting

The convection-diffusion equation

$$\nabla \cdot (f_c(w) - f_v(w, \nabla w)) = s(w, \nabla w)$$

can be written as a first order system

$$q = \nabla w$$

$$\nabla \cdot (f_c(w) - f_v(w, q)) = s(w, q)$$
Discretization

Find \((q_h, w_h) \in (V_h, W_h)\) s.t. \(\forall (\tau_h, \varphi_h) \in (V_h, W_h)\)

\[
0 = (\tau_h, q_h)_{\mathcal{T}_h} + (\nabla \cdot \tau_h, w_h)_{\mathcal{T}_h} - \langle \tau_h, \hat{w}_h \rangle_{\partial \mathcal{T}_h}
\]
Discretization

Find \((q_h, w_h) \in (V_h, W_h)\) s.t. \(\forall (\tau_h, \varphi_h) \in (V_h, W_h)\)

\[\begin{align*}
0 &= (\tau_h, q_h)_{T_h} + (\nabla \cdot \tau_h, w_h)_{T_h} - \langle \tau_h, \hat{w}_h \rangle_{\partial T_h} \\
0 &= - (\nabla \varphi_h, f_c(w_h) - f_v(w_h, q_h))_{T_h} - (\varphi_h, s(w_h, q_h))_{T_h} + \langle \varphi_h, \hat{f}_c - \hat{f}_v \rangle_{\partial T_h}
\end{align*}\]
Discretization

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\]

where

\[
V_h = \{ v \in L^2(\Omega)^d : v|_{\Omega_k} \in P^p(\Omega_k)^d, \Omega_k \in T_h \}
\]

\[
W_h = \{ w \in L^2(\Omega) : w|_{\Omega_k} \in P^p(\Omega_k), \Omega_k \in T_h \}
\]
Discretization — Introduction of $\lambda$

Find $(q_h, w_h, \lambda_h) \in (V_h, W_h, M_h)$ s.t. $\forall (\tau_h, \varphi_h, \mu_h) \in (V_h, W_h, M_h)$

$0 = \mathcal{N}_h (q_h, w_h, \lambda_h; \tau_h, \varphi_h, \mu_h)$
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$$+ \langle \mu_h, \left[ \hat{f}_c - \hat{f}_v \right] \rangle_{\Gamma_h}$$
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and

\[ \hat{f}_c (\lambda_h, w_h) = f_c (\lambda_h) - \alpha_c (\lambda_h - w_h) \]
\[ \hat{f}_v (\lambda_h, w_h, q_h) = f_v (\lambda_h, q_h) - \alpha_v (\lambda_h - w_h) \]
The linearized global system

\[
\begin{bmatrix}
A & B & R \\
C & D & S \\
L & M & N
\end{bmatrix}
\begin{bmatrix}
\delta Q \\
\delta W \\
\delta \Lambda
\end{bmatrix}
=
\begin{bmatrix}
F \\
G \\
H
\end{bmatrix}
\]
Hybridization

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\begin{bmatrix}
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G
\end{bmatrix} -
\begin{bmatrix}
R \\
S
\end{bmatrix} \delta \Lambda
\]

and

\[
L\delta Q + M\delta W + N\delta \Lambda = H.
\]
Substituting the first into the second equation yields the hybridized system

\[
\left( N - [L, M] \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} R \\ S \end{bmatrix} \right) \delta \Lambda = H - [L, M] \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} F \\ G \end{bmatrix}
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- The matrix \[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}
\] is block-diagonal such that the local problems can be solved element wise.
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\]

- The matrix \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) is block-diagonal such that the local problems can be solved element wise.
- The global hybridized system is formulated in terms of \( \delta \Lambda \) only and thus considerably smaller than the usual global system.
Adjoint-based Error Estimation

We are interested in quantifying the error of the computed target functional, i.e.

\[ e_h := J(x) - J(x_h) = J'[x_h](x - x_h) + O(\|x - x_h\|^2) \]

with the primal solution \( x_h = (q_h, w_h, \lambda_h) \).
Adjoint-based Error Estimation

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The link between variations in the residual and in the target functional is given by the so-called adjoint equation

\[ N_h'[x_h](dx_h; z_h) = J'[x_h](dx_h) \]

with the dual solution \( z_h = (\tilde{q}_h, \tilde{w}_h, \tilde{\lambda}_h) \).
Adjoint-based Error Estimation

We are interested in quantifying the error of the computed target functional, i.e.

\[ e_h := J(x) - J(x_h) = J'(x_h)(x - x_h) + O(\|x - x_h\|^2) \]

with the primal solution \( x_h = (q_h, w_h, \lambda_h) \).

The link between variations in the residual and in the target functional is given by the so-called adjoint equation

\[ \mathcal{N}'_h [x_h] (dx_h; z_h) = J' [x_h] (dx_h) \]

with the dual solution \( z_h = (\tilde{q}_h, \tilde{w}_h, \tilde{\lambda}_h) \).

Now, one can estimate the error by

\[ e_h \approx \mathcal{N}_h (x_h; z_h) \]
Adjoint-based Error Estimation

In matrix form, the adjoint equation reads as follow

\[
\begin{bmatrix}
A & B & R \\
C & D & S \\
L & M & N
\end{bmatrix}^T \begin{bmatrix}
\tilde{Q} \\
\tilde{W} \\
\tilde{\Lambda}
\end{bmatrix} = \begin{bmatrix}
\tilde{F} \\
\tilde{G} \\
\tilde{H}
\end{bmatrix}
\]
Adjoint-based Error Estimation

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\end{bmatrix}^T
\begin{bmatrix}
\tilde{Q} \\
\tilde{W} \\
\tilde{\Lambda}
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{F} \\
\tilde{G} \\
\tilde{H}
\end{bmatrix}
\]

Using again static condensation, one obtains

\[
\left(N - [L, M] \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1} \begin{bmatrix}
R \\
S
\end{bmatrix}\right)^T \tilde{\Lambda} = \tilde{H} - [R^T, S^T] \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-T} \begin{bmatrix}
\tilde{F} \\
\tilde{G}
\end{bmatrix}
\]
Pure Convection

\[ \nabla \cdot (w, w) = 0 \quad (x, y) \in \Omega = [0, 1]^2 \]

\[ w(x, y) = \sin (x - y) \quad (x, y) \in \Gamma_{\text{in}} = \{(x, y) : x \cdot y = 0\} \]

The target functional computes the outflow over a part of the boundary, i.e.

\[ J(\lambda) = \int_{0}^{0.5} \lambda(x, 1) \, dx \]
Pure Convection
Pure Convection \((\rho = 2)\)
\[ \nabla \cdot (w, w) - \epsilon \Delta w = s \quad (x, y) \in \Omega = [0, 1]^2 \]
\[ w(x, y) = 0 \quad (x, y) \in \partial \Omega \]

We set \( s \) such that
\[ w(x, y) = \left( x + \frac{e^{x/\epsilon} - 1}{1 - e^{1/\epsilon}} \right) \cdot \left( y + \frac{e^{y/\epsilon} - 1}{1 - e^{1/\epsilon}} \right), \epsilon = 0.01 \]

is a solution to the equation.

The target functional of interest is the mean value, i.e.
\[ J(w) = \int_{\Omega} w(x, y) \, dx \]
“Confusion”
“Confusion” ($\rho = 2$)
Inviscid Flow over a Smooth Bump

\[ J(w) = \sqrt{\frac{1}{|\Omega|} \int_{\Omega} \left( \frac{p/\rho^\gamma - p_\infty/\rho_\infty}{p_\infty/\rho^\gamma} \right)^2 \, dx} \]
Inviscid Flow over a Smooth Bump ($\rho = 3$)
Subsonic Flow around the NACA0012 Airfoil

$Ma = 0.5$, $\alpha = 2^\circ$, $J(\lambda) = c_D (\lambda)$
Subsonic Flow around the NACA0012 Airfoil

![Graph showing the comparison of Adjoint, Corrected, Residual, and Reference values against the parameter h=1/sqrt(ndof)]
Subsonic Flow around the NACA0012 Airfoil ($\rho = 2$)
Transonic Flow around the NACA0012 Airfoil

\[ \text{Ma} = 0.8, \ \alpha = 1.25^\circ, \ J(\lambda) = c_D(\lambda) \]
Transonic Flow around the NACA0012 Airfoil ($\rho = 2$)
Transonic Flow around the NACA0012 Airfoil ($p = 2$)
Ma = 0.5, \( \alpha = 1^\circ \), Re = 5000, \( J(\lambda, q) = c_D (\lambda, q) \)
Viscous Flow around the NACA0012 Airfoil ($\rho = 2$)
Viscous Flow around the NACA0012 Airfoil \((p = 2)\)
Conclusions

- Reducing degrees of freedom globally via hybridization
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- Efficient distribution of degrees of freedoms via adjoint-based adaptivity
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Future work:
- Full hp-adaptivity
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- Efficient distribution of degrees of freedoms via adjoint-based adaptivity
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Future work:
- Full hp-adaptivity
- Turbulent flow
Conclusions

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- Efficient distribution of degrees of freedoms via adjoint-based adaptivity
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Future work:
- Full hp-adaptivity
- Turbulent flow
- Parallelism
Acknowledgement

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Boundary Conditions

Find \((q_h, w_h, \lambda_h) \in (V_h, W_h, M_h)\) s.t. \(\forall (\tau_h, \varphi_h, \mu_h) \in (V_h, W_h, M_h)\)

\[
0 = (\tau_h, q_h)_{\mathcal{T}_h} + (\nabla \cdot \tau_h, w_h)_{\mathcal{T}_h} - \langle \tau_h, \lambda_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} - \langle \tau_h, w_{\partial \Omega} (\lambda_h) \rangle_{\partial \mathcal{T}_h \cap \partial \Omega}
\]

\[
0 = - (\nabla \varphi_h, f_c(w_h) - f_v(w_h, q_h))_{\mathcal{T}_h} - (\varphi_h, s(w_h, q_h))_{\mathcal{T}_h} + \langle \varphi_h, \hat{f}_c - \hat{f}_v \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega}
+ \langle \varphi_h, n \cdot (f_c (w_{\partial \Omega} (\lambda_h)) - f_v (\lambda_h, q_h)) - (\lambda_h - w_{\partial \Omega} (w_h)) \rangle_{\partial \mathcal{T}_h \cap \partial \Omega}
\]

\[
0 = \langle \mu_h, \begin{bmatrix} \hat{f}_c - \hat{f}_v \end{bmatrix} \rangle_{\Gamma_h \setminus \partial \Omega}
+ \langle \mu_h, n \cdot (f_v (\lambda_h, q_h) - f_v, \partial \Omega (f_v (w_{\partial \Omega} (\lambda_h), q_h)) - (\lambda_h - w_{\partial \Omega} (w_h)) \rangle_{\Gamma_h \cap \partial \Omega}
\]