A Hybridized Discontinuous Galerkin Method for Turbulent Compressible Flow

Michael Woopen, Thomas Ludescher, and Georg May

Graduate School AICES, RWTH Aachen University

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Motivation

▶ Turbulent flows are everywhere!
▶ High-order methods
  ▶ More accurate than state-of-the-art finite volume methods
  ▶ (potentially) more efficient
▶ Implicit methods → Stable, but high memory requirements
▶ Hybridization → Reduce system size
Hybridization?
Hybridization

Remove coupling between elements
Hybridization

Establish coupling between element-interfaces
The convection-diffusion equation

\[ \nabla \cdot (f_c(w) - f_v(w, \nabla w)) = s(w, \nabla w) \]

can be written as a first order system

\[ q = \nabla w \]

\[ \nabla \cdot (f_c(w) - f_v(w, q)) = s(w, q) \]
Discretization

Find \((q_h, w_h) \in (V_h, W_h)\) s.t. \(\forall (\tau_h, \varphi_h) \in (V_h, W_h)\)

\[
0 = N_h^{DG} (q_h, w_h; \tau_h, \varphi_h, ) = (\tau_h, q_h)_{T_h} + (\nabla \cdot \tau_h, w_h)_{T_h} - \langle n \cdot \tau_h, \hat{w}_h \rangle_{\partial T_h} - (\nabla \varphi_h, f_c(w_h) - f_v(w_h, q_h))_{T_h} - (\varphi_h, s(w_h, q_h))_{T_h} + \langle \varphi_h, \hat{f}_c - \hat{f}_v \rangle_{\partial T_h}
\]

where

\[
V_h = \{ v \in L^2(\Omega)^d : v|_T \in \Pi^p(T)^d, T \in T_h \}
\]

\[
W_h = \{ w \in L^2(\Omega) : w|_T \in \Pi^p(T), T \in T_h \}
\]
Discretization — Introduction of $\lambda$

Find $(q_h, w_h, \lambda_h) \in (V_h, W_h, M_h)$ s.t. $\forall (\tau_h, \varphi_h, \mu_h) \in (V_h, W_h, M_h)$

\[ 0 = \mathcal{N}_h(q_h, w_h, \lambda_h; \tau_h, \varphi_h, \mu_h) \]

\[ := (\tau_h, q_h)_{T_h} + (\nabla \cdot \tau_h, w_h)_{T_h} - \langle \mathbf{n} \cdot \tau_h, \lambda_h \rangle_{\partial T_h} \]

\[- (\nabla \varphi_h, f_c(w_h) - f_v(w_h, q_h))_{T_h} - (\varphi_h, s(w_h, q_h))_{T_h} + \langle \varphi_h, \hat{f}_c - \hat{f}_v \rangle_{\partial T_h} \]

\[+ \langle \mu_h, [\hat{f}_c - \hat{f}_v] \rangle_{\partial T_h} \]

where

\[ V_h = \{ v \in L^2(\Omega)^d : v|_T \in \Pi^p(T)^d, T \in T_h \} \]

\[ W_h = \{ w \in L^2(\Omega) : w|_T \in \Pi^p(T), T \in T_h \} \]

\[ M_h = \{ \mu \in L^2(\Gamma_h) : \mu|_e \in \Pi^p(e), e \in \Gamma_h \} \]

and

\[ \hat{f}_c(\lambda_h, w_h) = f_c(\lambda_h) - S_c(\lambda_h)(\lambda_h - w_h) \]

\[ \hat{f}_v(\lambda_h, w_h, q_h) = f_v(\lambda_h, q_h) - S_v(\lambda_h)(\lambda_h - w_h) \]
Hybridization

The linearized global system

\[
\begin{bmatrix}
A & B & R \\
C & D & S \\
L & M & N
\end{bmatrix}
\begin{bmatrix}
\delta Q \\
\delta W \\
\delta \Lambda
\end{bmatrix} =
\begin{bmatrix}
F \\
G \\
H
\end{bmatrix}
\]

can be written as

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
\delta Q \\
\delta W
\end{bmatrix} =
\begin{bmatrix}
F \\
G
\end{bmatrix} -
\begin{bmatrix}
R \\
S
\end{bmatrix} \delta \Lambda
\]

and

\[
L\delta Q + M\delta W + N\delta \Lambda = H.
\]
Hybridization

Substituting the first into the second equation yields the hybridized system

$$
\left( N - [L, M] \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} R \\ S \end{bmatrix} \right) \delta \Lambda = H - [L, M] \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} F \\ G \end{bmatrix}
$$

- The matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is block-diagonal such that the local problems can be solved element-wise.
- The global hybridized system is formulated in terms of $\delta \Lambda$ only and thus considerably smaller than the usual global system.
RANS + $k$-$\omega$

$$\nabla \cdot (f_c(w) - f_v(w, \nabla w)) = s(w, \nabla w)$$

$$w = \left( \rho, \rho v^T, \rho E, \rho k, \rho \omega \right)^T.$$ 

$$f_c(w) = \begin{pmatrix} 
\rho v^T \\
\rho v \otimes v + (p + \frac{2}{3} \rho k) I_d \\
(\rho E + p - \frac{1}{3} \rho k) v^T \\
\rho k v^T \\
\rho \omega v^T 
\end{pmatrix}, \quad f_v(w, \nabla w) = \begin{pmatrix} 
0 \\
\tau \\
\tau v + \kappa \nabla T \\
(\mu + \sigma_k \mu_t) \nabla k \\
(\mu + \sigma_\omega \mu_t) \nabla \omega 
\end{pmatrix}$$

$$s(w, \nabla w) = \begin{pmatrix} 
0 \\
0 \\
-(\tau^R : \nabla v - \beta_k \rho k^* \omega_r) \\
\tau^R : \nabla v - \beta_k \rho k^* \omega_r \\
\frac{\alpha \omega \omega}{k} \tau^R : \nabla v - \beta_\omega \rho \omega^2 
\end{pmatrix}$$
\[ \nabla \cdot (f_c(w) - f_v(w, \nabla w)) = s(w, \nabla w) \]

\[ w = \left( \rho, \rho v^T, \rho E, \rho k, \rho \hat{\omega} \right)^T. \]

\[ f_c(w) = \begin{pmatrix} \rho v^T \\ \rho v \otimes v + \left( p + \frac{2}{3} \rho k \right) \text{Id} \\ (\rho E + p - \frac{1}{3} \rho k) v^T \\ \rho k v^T \\ \rho \hat{\omega} v^T \end{pmatrix}, \quad f_v(w, \nabla w) = \begin{pmatrix} 0 \\ \tau v + \kappa \nabla T \\ (\mu + \sigma_k \mu_t) \nabla k \\ (\mu + \sigma_\omega \mu_t) \nabla \hat{\omega} \end{pmatrix}, \]

\[ s(w, \nabla w) = \begin{pmatrix} 0 \\ 0 \\ - (\tau_R : \nabla v - \beta_k \rho k^* e^{\hat{\omega}_r}) \\ \tau_R^* : \nabla v - \beta_k \rho k^* e^{\hat{\omega}_r} \\ \frac{\alpha_w}{k} \tau_R^* : \nabla v - \beta_\omega \rho e^{\hat{\omega}_r} + (\mu + \sigma_\omega \mu_t) (\nabla \hat{\omega})^2 \end{pmatrix}. \]
Flat Plate

- $Ma = 0.2$
- $Re = 5 \cdot 10^6$
- $Tu = 4 \cdot 10^{-4}$
- $\mu_t/\mu = 9 \cdot 10^{-3}$
- $n_e = 816,3264$ (http://turbmodels.larc.nasa.gov/)
Flat Plate — $Ma = 0.2, \ Re = 5 \cdot 10^6$
Flat Plate — Streamwise Velocity ($n_e = 816$)
Flat Plate — Turbulent Kinetic Energy ($n_e = 816$)

The graph shows the distribution of turbulent kinetic energy ($k^+$) as a function of the wall-normal distance ($y^+$). There are three curves for different values of $p$:

- $p = 1$ (black line)
- $p = 2$ (green dashed line)
- $p = 3$ (blue line)

The x-axis represents $y^+$ on a logarithmic scale, while the y-axis represents $k^+$ also on a logarithmic scale.
Flat Plate — Turbulent Dissipation Rate \((n_e = 816)\)
Flat Plate — Eddy Viscosity Ratio ($n_e = 816$)
Flat Plate — Streamwise velocity ($n_e = 3264$)
Flat Plate — Turbulent Kinetic Energy ($n_e = 3264$)

\[ k^+ = \frac{k}{\nu} \]

\[ p = 1, 2, 3 \]

Graph showing the variation of $k^+$ with $y^+$ for different values of $p$. The graph illustrates the trend of $k^+$ as $y^+$ increases, reaching a peak and then decreasing.
Flat Plate — Turbulent Dissipation Rate ($n_e = 3264$)
Flat Plate — Eddy Viscosity Ratio ($n_e = 3264$)
Flat Plate — Skin Friction Coefficient

\[ C_f = \frac{1}{p} \]

The graph shows the skin friction coefficient \( C_f \) as a function of \( x \) for different values of \( p \):
- \( p = 1 \) (black line)
- \( p = 2 \) (green dashed line)
- \( p = 3 \) (blue dashed line)
- Theory (red dotted line)
Flat Plate — Skin Friction Coefficient

The graph shows the skin friction coefficient ($C_f$) as a function of the parameter $x$, for different values of $p$. The graph includes theoretical predictions and experimental data.

- Line with $p = 1$ represents the theoretical curve.
- Dashed line with $p = 2$ represents another theoretical curve.
- Dotted line with $p = 3$ represents yet another theoretical curve.
- Dotted-dashed line with Theory represents the experimental or calculated data.

The graph is plotted on a logarithmic scale, with $C_f$ on the y-axis ranging from $10^{-3}$ to $10^{-1}$, and $x$ on the x-axis ranging from $10^{-4}$ to $10^1$.
RAE 2822 Case 9

- \( Ma = 0.734 \)
- \( \alpha = 2.79^\circ \)
- \( Re = 6.5 \cdot 10^6 \)
- \( Tu = 4 \cdot 10^{-4} \)
- \( \mu_t/\mu = 9 \cdot 10^{-3} \)
- \( n_e = 4048 \) (http://www.dlr.de/as/hiocfd)
RAE 2822 — $Ma = 0.734$, $\alpha = 2.79^\circ$, $Re = 6.5 \cdot 10^6$
RAE 2822 — Mach number \((n_e = 4048)\)
RAE 2822 — Turbulent Kinetic Energy \((n_e = 4048)\)
RAE 2822 — Pressure Coefficient

Experiment

\[ p = 1, \; n_e = 4048 \]
\[ p = 2, \; n_e = 4048 \]
Conclusion

- Validated RANS-$k$-$\omega$ with HDG
- First results for subsonic and transonic flow

Outlook:

- Adjoint-based anisotropic mesh adaptation (next week @ICOSAHOM)
- Increase robustness of the non-linear solver
- Additional turbulence models (SA, $k$-$\omega$ SST)
Shock-Capturing

Artificial viscosity approach for shock-capturing:

\[ \nabla \cdot (\epsilon (\mathbf{w}, \nabla \mathbf{w}) \nabla \mathbf{w}) \] (1)

The element-wise constant artificial viscosity is defined by

\[ \epsilon_K := \frac{\epsilon_0 \hat{h}_K^{2-\beta}}{|K|} \int_K \sum_{i=1}^{m} |(\nabla \cdot \mathbf{f}_c(w))_i| \, \mathrm{d}x \]

where the effective mesh resolution is given by \( \hat{h}_K := \frac{h_K}{p_K} \).

The discretized term finally reads

\[ \mathcal{N}_{h,sc} (\mathbf{q}_h, \mathbf{w}_h; \varphi_h) := (\nabla \varphi_h, \epsilon (\mathbf{w}_h, \mathbf{q}_h) \mathbf{q}_h)_{T_h} \] (2)
Boundary Conditions

Use analytical fluxes with boundary states and gradients

\[ \mathcal{N}_{h, \partial \Omega} (q_h, w_h; \tau_h, \varphi_h) := \left\langle \tau_h \cdot n, w_{\partial \Omega} (w_h) \right\rangle_{\Gamma_b}^h + \left\langle \varphi_h, \left( f_c (w_{\partial \Omega}) - f_v (w_{\partial \Omega}, q_h, \partial \Omega) \right) \cdot n \right\rangle_{\Gamma_b}^h \]