

# Lyapunov-Based Error Bounds for the Reduced-Basis Method

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**Abstract:** We introduce two new error bounds for the reduced-basis method. Existing error bounds for parabolic problems can be classified as either space time or time stepping. Space-time bounds are much more costly and often become unpractical. The cheaper times-stepping bounds have always failed to adequately represent the dynamics of systems containing noncoercive operators. As a result they have always produced extremely pessimistic bounds. Our new bounds are time-stepping bounds that make use of the Lyapunov stability theory to better capture the dynamics of the system.

*Keywords:* Error Estimation, Reduced-Order Models, Lyapunov Function, Lyapunov Equation, Stability Analysis, Parameter-Dependent Systems, Dynamic Systems

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## 1. INTRODUCTION

Reduced-basis modeling is a powerful tool for approximating solutions of parameter-dependent PDEs. In recent years it has been applied to many types of PDEs allowing for drastic decreases in computational time. As with all model-reduction techniques, error quantification is of great importance. If the error is too large, the model will not be useful. Error bounds also play a second role in reduced-basis modeling; they are used in the construction of the model itself.

Two classical error bounds that can be used in conjunction with the reduced-basis method for parabolic equations are the energy-error estimate given by Grepl and Patera (2005), and the  $L^2$ -error bound given by Haasdonk and Ohlberger (2008). Both are mainly for use with coercive problems. If they are extended for use with noncoercive problems, they predict exponential growth of the error in time. In such cases they are of little practical use.

The problem is that these bounds are based on how the system evolves in a single time step. This is convenient because it allows us to take complete advantage of the evolutionary nature of the system but has resulted in very pessimistic error bounds because the norms that have been used fail to capture the nature of the dynamics. An alternative approach is to use bounds based on the space-time formulation given by Urban and Patera (2014). That has been shown to be very successful in producing more accurate error bounds but is associated with greatly increased computational costs. Such bounds fail to take advantage of the evolutionary nature of the system and require the calculation of stability constants for the full space-time system.

In this article we present new generalizations of the time-stepping bounds. Our new bounds take full advantage of the evolutionary nature of the problem making them much cheaper than space-time bounds. At the same time they

also model the dynamics of the system better so that the bounds do not grow exponentially. This is achieved using the Lyapunov stability theory and norms in which the error is well behaved.

For a large number of noncoercive problems our new bounds will produce better results than any previous bounds. They could also be highly useful in simulating closed-loop systems, which are often noncoercive.

## 2. PROBLEM SETUP AND ASSUMPTIONS

Most applications of our work will involve the approximation of PDEs but we will start directly with a semi-discrete system, which could be a discretization of a PDE. We will consider the problem of approximating the following system for any parameter  $\mu$  in a bounded, finite-dimensional domain  $\mathcal{D} \subset \mathbb{R}^p$ .

$$M(\mu)\dot{y}(t) + A(\mu)y(t) = B(\mu)u(t) \quad (1)$$

Here we have  $M(\mu) \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ ,  $A(\mu) \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ ,  $B(\mu) \in \mathbb{R}^{\mathcal{N} \times m}$ , the inputs  $u \in \mathbb{R}^m$ , and the state  $y(t) \in \mathbb{R}^{\mathcal{N}}$ . We will write  $\dot{y}$  to denote the time derivative of  $y$  and choose a symmetric positive definite (SPD) matrix  $X \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$  that we will use as an inner product. In section 6 we will assume that  $M(\mu)$  be symmetric but most of our results will hold even if it is not.

In order to approximate solutions to (1) numerically we will use a backwards-Euler time discretization with  $K$  time steps of uniform length  $\tau$  to get the following system

$$M(\mu)y^k + \tau A(\mu)y^k = M(\mu)y^{k-1} + \tau B(\mu)u^k, \quad (2)$$

for  $1 \leq k \leq K$  which will be referred to as our truth system. For simplicity we will assume that the initial state  $y^0$  is zero.

As is usual in the reduced-basis context, we will be interested in approximating  $y$  for any  $\mu$  in a bounded parameter

domain  $\mathcal{D}$ . In order to ensure that our method will be computationally efficient we require that the parameter-dependence of  $M$ ,  $A$ , and  $B$  be of the following form

$$G(\mu) = \sum_{q=1}^{Q_G} \Theta_G^q(\mu) G^q, \quad (3)$$

where  $G$  should be replaced by  $A$ ,  $M$  or  $B$  respectively. Here the parameter-independent components  $G^q$  will have the same size as the associated matrix  $G$  and the parameter maps  $\Theta_G^q(\cdot)$  can be arbitrary smooth functions.

In later sections we will often suppress the  $\mu$ -dependence of the matrices to save space and simplify the notation.

### 3. THE REDUCED-BASIS METHOD FOR PARABOLIC PROBLEMS

Following the work of Grepl and Patera (2005) we will use a Galerkin projection to reduce the truth model. We start by introducing a matrix  $Z \in \mathbb{R}^{N \times N}$ , where  $N \ll \mathcal{N}$ . The columns of  $Z$  will be our  $N$  basis elements and should be orthonormal with respect to  $X$ . We will approximate our truth model with the reduced-basis model

$$M_N(\mu) y_N^k + \tau A_N(\mu) y_N^k = M_N(\mu) y_N^{k-1} + \tau B_N(\mu) u^k, \quad (4)$$

for  $1 \leq k \leq K$ , where  $M_N := Z^T M Z$ ,  $A_N := Z^T A Z$ ,  $B_N := Z^T B$ , and  $y_N^0 = 0$ .

In previous time-stepping methods the coercivity constant

$$\alpha(\mu) := \inf_{v \neq 0} \frac{v^T A(\mu) v}{v^T X v} \quad (5)$$

played a vital role in both establishing the stability of reduced-basis approximations and error estimation. The operator  $A(\mu)$  is said to be coercive if  $\alpha(\mu) > 0$ . Grepl and Patera (2005) assume that both  $A(\mu)$  and  $M(\mu)$  are uniformly coercive and that  $M(\mu)$  is symmetric for all  $\mu \in \mathcal{D}$ . In that case it is easy to show that (4) is stable both numerically and in the sense of Lyapunov for all choices of  $Z$ . Our relaxed assumptions do not suffice to guarantee stability but in section 7 we discuss ways in which stability can be ensured.

#### 3.1 Building the Reduced Basis

Over the years many methods have been developed to build the reduced basis, onto which a system will be projected. In general the most effective method seems to be the POD-greedy method introduced by Haasdonk and Ohlberger (2008). The method is based on the greedy method that was used by Veroy et al. (2003) for stationary problems. In both cases the key idea is that a search is performed over a finite subset of  $\mathcal{D}$  to find parameter values for which the model produces large error bounds. The model can then be improved using the truth solution associated with those parameter values.

In some cases the input  $u$  to the system will be parameter dependent or constant. In that case the input can be handled directly by the POD-greedy algorithm. If that is not the case, it may be useful to handle the input using impulse responses as done by Grepl and Patera (2005).

For our work the traditional methods will in many cases suffice but this is not guaranteed. Even if the truth system is stable, the reduced model can have stability issues. Such issues will be discussed in section 7.

## 4. LYAPUNOV STABILITY FOR LTI SYSTEMS

We will now introduce Lyapunov stability, which, as we will show, can be used to take advantage of the structure of the dynamical system and build cheap and effective error bounds.

### 4.1 Continuous-Time Systems

For continuous-time systems we have the following classical theorem.

*Theorem 1.* Given a fixed SPD matrix  $P \in \mathbb{R}^{N \times N}$  the function  $V(v) = v^T M(\mu)^T P M(\mu) v$  is a Lyapunov function for (1) at the parameter value  $\mu \in \mathcal{D}$ , iff the symmetric matrix

$$Q(\mu) := \frac{A(\mu)^T P M(\mu) + M(\mu)^T P A(\mu)}{2} \quad (6)$$

is positive definite. In that case the system is asymptotically stable for  $\mu$ .

A sufficient condition to show that (1) is asymptotically stable for a given parameter value  $\mu \in \mathcal{D}$  is that the coercivity constant

$$\alpha_Q(\mu) := \inf_{v \neq 0} \frac{v^T Q(\mu) v}{v^T X v} \quad (7)$$

be positive. In that case we will say that  $Q(\mu)$  is coercive.

### 4.2 Discrete-Time Systems

For the discrete-time system (2) we have the following theorem.

*Theorem 2.* Given a fixed SPD matrix  $P \in \mathbb{R}^{N \times N}$  the function  $V_D(v) = v^T (M(\mu) + \tau A(\mu))^T P (M(\mu) + \tau A(\mu)) v$  is a Lyapunov function for (2) at the parameter value  $\mu \in \mathcal{D}$ , iff the symmetric matrix

$$Q_D(\mu) := \frac{A^T P M + M^T P A + \tau A^T P A}{2} \quad (8)$$

is positive definite. In that case the system is asymptotically stable for  $\mu$ .

We note that  $Q_D - Q$  is semi-positive definite. That implies that any  $P$  that proves the stability of (1) also proves the stability of (2). For  $Q_D$  we introduce the coercivity constant  $\alpha_{Q_D}$ , which is defined analogous to  $\alpha_Q$ .

## 5. LYAPUNOV-BASED ERROR BOUNDS

In this section we derive generalized versions of the energy and the  $L_2$  error bounds using Lyapunov stability theory. In both cases the error that we wish to measure will be given by  $e^k = y^k - Z y_N^k$ . We note that  $y_N^k$  does not need to be a reduced-basis approximation and could be any lower-dimensional approximation of  $y^k$ .

### 5.1 Generalized Energy Error Bound

In order to derive a generalized version of the energy error bound we will consider Lyapunov stability for the continuous-time system (1). We assume that we have an SPD matrix  $P \in \mathbb{R}^{N \times N}$  such that  $V(v) = v^T M(\bar{\mu})^T P M(\bar{\mu}) v$  is a Lyapunov function for the system (1) for a particular  $\bar{\mu} \in \mathcal{D}$ .

The issue of finding a matrix  $P$  that satisfies the requirements will be considered in section 6. We fix  $P$  but allow  $A$ ,  $M$ , and  $B$  to be functions of the parameter. As a result  $Q$  and  $Q_D$  will also be parameter dependent. Assuming that the parameter dependences are smooth,  $V(v)$  will be a Lyapunov function for all  $\mu$  in some neighborhood of  $\bar{\mu}$  despite the fact that  $P$  is fixed.

We now introduce the usual residual

$$r^k := Bu^k - \frac{1}{\tau} M Z y_N^k - A Z y_N^k + \frac{1}{\tau} M Z y_N^{k-1}, \quad (9)$$

and a new parameter-dependent norm

$$\|v\|_{P,\mu} := \sqrt{(v^K)^T M^T P M v^K + \tau \sum_{k=1}^K (v^k)^T Q v^k}, \quad (10)$$

which we will call the generalized energy norm. We note that it depends on both  $P$  and  $\mu$ .

*Theorem 3.* The error in the generalized energy norm satisfies

$$\|e\|_{P,\mu} \leq \Delta_P(\mu) \quad (11)$$

where

$$\Delta_P(\mu) := \sqrt{\frac{\tau}{\alpha_Q(\mu)} \sum_{k=1}^K (r^k)^T P M X^{-1} M^T P r^k}, \quad (12)$$

for all  $\mu$  for which  $\alpha_Q(\mu) > 0$ .

**Proof.**

Combining (2) and (9) gives

$$M e^k + \tau A e^k = M e^{k-1} + \tau r^k. \quad (13)$$

We then multiply by  $(e^k)^T M^T P$  on the left-hand side.

$$\begin{aligned} (e^k)^T M^T P M e^k + \tau (e^k)^T M^T P A e^k \\ = (e^k)^T M^T P M e^{k-1} + \tau (e^k)^T M^T P r^k \end{aligned} \quad (14)$$

The second term in (14) can be rewritten using the fact that  $Q$  is the symmetric part of  $M^T P A$ .

$$(e^k)^T M^T P A e^k = (e^k)^T Q e^k \quad (15)$$

The third term in (14) can be bounded as follows.

$$\begin{aligned} (e^k)^T M^T P M e^{k-1} \\ \leq ((e^k)^T M^T P M e^k)^{\frac{1}{2}} ((e^{k-1})^T M^T P M e^{k-1})^{\frac{1}{2}} \\ \leq \frac{1}{2} (e^k)^T M^T P M e^k + \frac{1}{2} (e^{k-1})^T M^T P M e^{k-1} \end{aligned} \quad (16)$$

We use similar methods and the coercivity of  $Q(\mu)$  to get

$$\begin{aligned} \tau (e^k)^T M^T P r^k \\ \leq \tau ((e^k)^T X e^k)^{\frac{1}{2}} ((r^k)^T P M X^{-1} M^T P r^k)^{\frac{1}{2}} \\ \leq \tau ((e^k)^T Q e^k)^{\frac{1}{2}} \left( \frac{(r^k)^T P M X^{-1} M^T P r^k}{\alpha_Q(\mu)} \right)^{\frac{1}{2}} \\ \leq \frac{\tau}{2} (e^k)^T Q e^k + \frac{\tau}{2} \frac{(r^k)^T P M X^{-1} M^T P r^k}{\alpha_Q(\mu)}. \end{aligned} \quad (17)$$

The previous results now allow us to rewrite (14) as

$$\begin{aligned} (e^k)^T M^T P M e^k - (e^{k-1})^T M^T P M e^{k-1} \\ + \tau (e^k)^T Q(\mu) e^k \leq \tau \frac{(r^k)^T P M X^{-1} M^T P r^k}{\alpha_Q(\mu)}. \end{aligned} \quad (18)$$

Summing (18) over  $1 \leq k \leq K$  with  $e^0 = 0$  gives

$$\begin{aligned} (e^K)^T M^T P M e^K + \tau \sum_{k=1}^K (e^k)^T Q(\mu) e^k \\ \leq \frac{\tau}{\alpha_Q(\mu)} \sum_{k=1}^K (r^k)^T P M X^{-1} M^T P r^k. \end{aligned} \quad (19)$$

Taking the square root of both sides finishes the proof.  $\square$

We note that, if we assume that  $M$  is SPD and that  $A$  is coercive then we can choose  $Q = (A + A^T)/2$  and  $P = M^{-1}$ . Our bound then reduces to the energy bound given by Grepl and Patera (2005) for coercive problems.

### 5.2 Generalized $L_2$ Error Bound

We now present a generalized version of the bound given by Haasdonk and Ohlberger (2008). In this case it will be more convenient to consider the Lyapunov theory that was given in section 4.2 for discrete-time systems. Let  $P \in \mathbb{R}^{N \times N}$  be an SPD matrix such that

$$V_D(v) = v^T (M + \tau A)^T P (M + \tau A) v \quad (20)$$

is a Lyapunov function for the system given in (2). We define the norms  $\|\cdot\|_{L,\mu}$  and  $\|\cdot\|_P$  such that  $\|v\|_{L,\mu}^2 := \sqrt{V_D(v)}$  and  $\|v\|_P := \sqrt{v^T P v}$  for all  $v \in \mathbb{R}^N$ . Assuming as before that  $e^0 = 0$  we have the following error bound.

*Theorem 4.* If the system in (2) is Lyapunov stable, the error at the  $k^{\text{th}}$  time step must satisfy

$$\|e^k\|_{L,\mu} \leq \tau \sum_{\kappa=1}^k C^{k-\kappa} \|r^\kappa\|_P, \quad (21)$$

where

$$C := \sup_{v \neq 0} \frac{\|M v\|_P}{\|v\|_{L,\mu}} < 1. \quad (22)$$

**Proof.** We begin by taking the  $P$ -norm of both sides of (13) and applying the triangle inequality to the right-hand side.

$$\|e^k\|_{L,\mu} \leq \|M e^{k-1}\|_P + \tau \|r^k\|_P \quad (23)$$

We then bound the second term in the last equation using (22) to get

$$\|e^k\|_{L,\mu} \leq C \|e^{k-1}\|_{L,\mu} + \tau \|r^k\|_P. \quad (24)$$

Equation (21) can then be derived iteratively starting with  $e^0 = 0$ . To see that  $C \leq 1$  we use (8) to get

$$\begin{aligned} C^2 &= \sup_{v \neq 0} \frac{\|Mv\|_P^2}{\|v\|_{L,\mu}^2} \\ &= \sup_{v \neq 0} \frac{v^T M^T P M v}{v^T (M^T P M + 2\tau Q_D) v} \\ &= 1 - 2\tau \inf_{v \neq 0} \frac{v^T Q_D v}{\|v\|_{L,\mu}^2} < 1. \end{aligned} \quad (25)$$

This expression is less than 1 due to the fact that the matrix  $Q_D$  is coercive.  $\square$

## 6. EFFICIENTLY COMPUTING LYAPUNOV FUNCTIONS

In this section we will consider the construction of the matrix  $P$  and introduce three possibilities, which we will denote  $P_1$ ,  $P_2$  and  $P_3$ . For the first we will use a classical theorem.

*Theorem 5.* For any SPD matrix  $Y \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$  there exists an SPD matrix  $P \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$  that solves the Lyapunov equation

$$Y = \frac{A(\bar{\mu})^T P M(\bar{\mu}) + M(\bar{\mu})^T P A(\bar{\mu})}{2}, \quad (26)$$

iff the system (1) is asymptotically stable for the given parameter value  $\bar{\mu} \in \mathcal{D}$ . In that case  $P$  is also the unique SPD solution and the quadratic function  $V(y) := y^T M^T P M y$  is a Lyapunov function for the parameter value  $\bar{\mu}$ .

If  $\mathcal{N}$  is small we can choose any SPD matrix  $Y_1$  and solve (26) to find a matrix  $P_1$  that will generate a Lyapunov function. Unfortunately, we are mostly interested in situations where  $\mathcal{N}$  is very large. In such cases computing and storing the  $(\mathcal{N}^2 + \mathcal{N})/2$  independent elements that make up the symmetric matrix  $P_1$  can be much too costly.

In order to derive an alternative to  $P_1$  let us assume that  $M(\bar{\mu})$  is symmetric and coercive. If  $A$  were also coercive, we could simply set  $P = M(\bar{\mu})^{-1}$  and get  $Y = A_S(\bar{\mu})$ . Here  $A_S(\bar{\mu}) := (A(\bar{\mu}) + A(\bar{\mu})^T)/2$  is the symmetric part of  $A$ . That would be very convenient because  $P$  is then the inverse of a sparse matrix. If  $A(\bar{\mu})$  is noncoercive, making small changes may suffice.

The idea is to change the noncoercive part of  $A_S$  to create a coercive matrix  $Y_2$ . Let us assume that we have two matrices  $R \in \mathbb{R}^{\ell \times \ell}$  and  $W \in \mathbb{R}^{\mathcal{N} \times \ell}$  with  $\ell \ll \mathcal{N}$  such that

$$Y_2 := A_S(\bar{\mu}) + X W R W^T X \quad (27)$$

is coercive. We then solve the Lyapunov equation

$$X W R W^T X = \frac{A(\bar{\mu})^T P_L M(\bar{\mu}) + M(\bar{\mu})^T P_L A(\bar{\mu})}{2} \quad (28)$$

to get the matrix  $P_L$ . The matrix  $P_L$  is not positive definite and thus cannot generate a Lyapunov equation. The matrix  $P_2 := M(\bar{\mu})^{-1} + P_L$ , on the other hand, is positive definite. To see that  $P_2$  can also generate a Lyapunov function, we use the linearity of the Lyapunov equation to verify that

$$Y_2 = \frac{A(\bar{\mu})^T P_2 M(\bar{\mu}) + M(\bar{\mu})^T P_2 A(\bar{\mu})}{2}. \quad (29)$$

It remains to show that  $W$  and  $R$  can be chosen such that  $Y_2$  is positive definite. We consider the eigenvalue problem  $A_S(\bar{\mu})v = \lambda X v$  and find all of the eigenvalues that are smaller than some positive constant  $\varepsilon$ . These are the eigenvalues that we wish to change and the associated eigenvectors will be the columns of  $W$ . They are chosen to be orthogonal to both  $X$  and  $A_S$  as well as normalized with respect to  $X$ . We can then set  $R = \sigma I - W^T A_S W$  for some  $\sigma \geq \varepsilon$ . This choice of  $W$  and  $R$  ensures that both  $Y_2$  and  $Y_2 - \varepsilon X$  are positive definite.

The matrix  $P_2$  is still too costly to use in practice but it can be very cheaply approximated. The idea is that rather than calculating  $P_L$  we can calculate a low-rank approximation to it. Such an approximation is given by  $G^T G$  where  $G \in \mathbb{R}^{s \times \mathcal{N}}$  with  $s \ll \mathcal{N}$ . This is generally possible if  $\ell \ll \mathcal{N}$ . Such approximations have been studied in great detail and many computational methods exist, including those by Li and White (2002); Gugercin et al. (2003); Benner and Saak (2013).

We now define  $P_3 := M(\bar{\mu})^{-1} + G^T G$ . Computing the actual matrix  $P_3$  is still very expensive. Instead we only calculate and store the  $s\mathcal{N}$  elements of the low-rank term  $G$ . Matrix-vector products can then be computed in a very efficient manner; they only require one sparse matrix solve and two very cheap matrix-vector products with  $G$  and  $G^T$ . Using  $P_3$  will ensure that our method remains cost effective even for very large values of  $\mathcal{N}$ .

## 7. STABILITY OF THE REDUCED-BASIS MODEL

Given the relaxed requirements on the dynamical system that we are considering, it is possible that for certain choices of the reduced basis the reduced model can be unstable either numerically or in the sense of Lyapunov despite the fact that the truth model is stable.

Numerical instability of reduced-systems is a topic of great interest in the area of reduced-basis modeling but has mostly been considered in the context of stationary problems. In the case of stationary problems that are coercive, the stability of the truth system generally guarantees the stability of the reduced model; however, if the truth system is only inf-sup stable then it can happen that reduced-models become unstable. A general method for dealing with such a problem is the use of Petrov-Galerkin methods in which the test space is systematically built using the double greedy method presented by Dahmen et al. (2014).

Similar, problems might occur in our case. If  $M_N(\mu) + \tau A_N(\mu)$  becomes nearly singular, it may be necessary to apply a sort of modified double greedy algorithm to create a more stable approximation.

Being that we are dealing with dynamical systems we also have to worry about Lyapunov stability. In reality, instability in the sense of Lyapunov will also induce another type of numerical instability. The error will grow exponentially in time, despite the fact that at each time step the computations might be well conditioned. Particularly for long time iterations this becomes a huge problem and is also the reason why reduced-basis modeling of systems that are not Lyapunov stable is of little practical use.

Problems with Lyapunov stability are different than the usual problems with numerical stability. With Lyapunov stability the problem is not generally in the choice of the test space but rather in the choice of the trial space. Let us assume that (2) is stable and  $M$  is SPD but that  $A$  is noncoercive for a given  $\mu$ . We choose  $Z \in \mathbb{R}^{N \times 1}$  such that  $Z^T A(\mu) Z \leq 0$ . The reduced system is then unstable in the Lyapunov sense. Nevertheless, the inf-sup constant of  $A_N(\mu)$  might be bounded away from zero. In such cases the problem cannot be solved by adding test functions like the double greedy algorithm would do. The problem is rather that stabilizing modes that are present in the truth system are missing in the reduced system. In this case the trial space needs to be enlarged.

In many cases the Lyapunov instability may be fixed by simply adding a few more POD modes to the initial basis. Once the basis is sufficiently large there should be no problem. Otherwise, the greedy algorithm will identify instabilities because they will cause large errors. It should then add new basis elements for those parameter values until the reduced system is stable.

In general it will be of interest to prove that both the reduced and truth models are stable for all parameter values. That is discussed briefly in section (9).

## 8. OFFLINE-ONLINE DECOMPOSITION

An important part of any reduced-basis method is the offline-online decomposition. The idea is to divide the workload into two phases. During the offline phase all computations with costs that depend on  $\mathcal{N}$  should be performed. The online phase should be kept as simple as possible and, in particular, independent of  $\mathcal{N}$ .

Offline-online decompositions are possible mainly due to decompositions of the form (3). For our system the offline-online decomposition is mostly the same as that given by Grepl and Patera (2005) except that more recent results exist for bounding of stability constants. They will be discussed in the next section.

## 9. BOUNDING STABILITY CONSTANTS

Stability constants are needed in two stages of our modeling process: error bounding and ensuring the well-posedness of our systems. Bounding the stability constants cannot be done entirely in the online stage because the necessary computations are dependent on  $\mathcal{N}$ . The common solution is to divide the necessary work using the successive constraint method (SCM), which has been discussed by Huynh et al. (2007); Chen et al. (2009). SCM can be used to bound  $\alpha$ ,  $\alpha_Q$ ,  $\alpha_{Q_D}$ , and  $C$  online. That enables us to evaluate error bounds and verify the stability of the system in an a posteriori fashion.

Unfortunately, SCM is largely inadequate to verify the system's stability. That should be done in the offline stage. The difficulty that has prevented that from being done in the past is that one would need to prove stability for all  $\mu \in \mathcal{D}$  and not just for discrete points in  $\mathcal{D}$  like one can do with SCM. Recent work by O'Connor (2016), which takes advantage of a certain concavity of coercivity constants, has now made that possible. The same ideas can also be

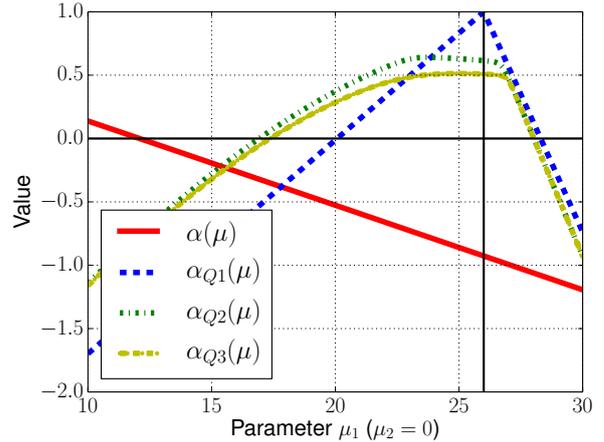


Fig. 1. Plot of the various coercivity constants over  $\mu_1$  used to speed up the evaluation of error bounds in the online stage.

## 10. NUMERICAL RESULTS

In this section we will consider an approximation to a diffusion-convection-reaction problem with the differential operator given by

$$A_\epsilon(\mu) := -\Delta + \mu_1 \left( x - \frac{1}{2} \right) \nabla + \mu_2, \quad (30)$$

where  $x$  is in the spatial domain  $\Omega = (0, 1)$ . For boundary conditions we will consider a zero-Dirichlet boundary at 0 and a harmonic Neumann boundary at 1. The system will have one input given by a uniform source term between  $1/6$  and  $2/6$ . Our spatial discretization for the truth model will be a piecewise linear finite-element approximation and have  $\mathcal{N} = 180$  degrees of freedom. Our parameter will be given by  $\mu = [\mu_1, \mu_2]^T$ , and we will set  $X = A([0, 1]^T)$  and  $\bar{\mu} = [26, 0]^T$ .

### 10.1 Constructing $P$

In section 6 we introduced three matrices that could be used to generate Lyapunov functions. Although only  $P_3$  should be used in applications, we will use all three to demonstrate our results. The problem specific details of their construction are discussed here.

- $P_1$ ) In order to determine  $P_1$  we will choose  $Y_1 = X$ .
- $P_2$ ) To design  $P_2$  we will use the values  $\epsilon = .5$  and  $\sigma = 1$ . The smallest eigenvalues of  $A_S$  with respect to  $X$  are  $-0.9267$  and  $0.6147$ . That means that we only need to change one eigenvalue and  $\ell = 1$ .  $W$  is then the eigenvector associated with the negative eigenvalue and  $R = [1.9267]$ .
- $P_3$ ) To build  $P_3$  we need to approximate  $P_L$ . The largest eigenvalues of  $P_L$  with respect to  $X$  are  $1.947$ ,  $5.510 \times 10^{-3}$ ,  $6.735 \times 10^{-4}$ , and  $9.227 \times 10^{-5}$  meaning that an approximation with a rank of two or three should suffice. We will use rank  $s = 2$ .

To be able to plot the coercivity we will fix  $\mu_2 = 0$ . Figure 1 shows the coercivity of the operator  $A$  as well as the coercivity constants  $\alpha_{Q_i}$  associated with the matrices  $P_i$  over a range of values for  $\mu_1$ . For  $\mu_1 > 14$  the operator  $A(\mu)$  is noncoercive. Thus in this region energy and  $L_2$

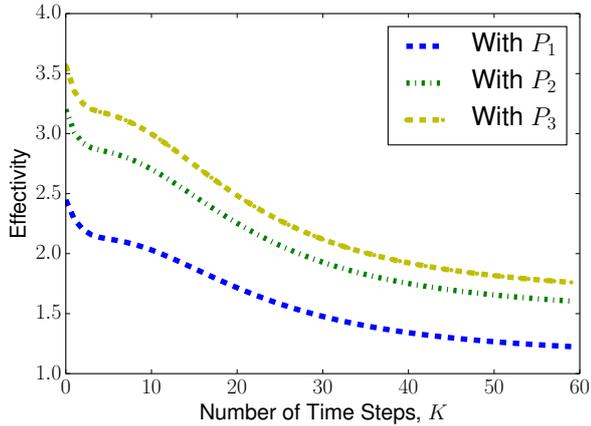


Fig. 2. Effectivity of the generalized energy bounds

bounds will expect exponential growth of the error despite the fact that the system is Lyapunov stable.

As we would have hoped the difference between  $\alpha_{Q2}$  and  $\alpha_{Q3}$  is minimal, suggesting that our rank-two approximation is sufficient. We also see that for  $\mu_1 \in [17.15, 28.02]$  the matrix  $P_3$  generates a Lyapunov function. That proves that the system is stable and that using  $P_3$  our error bounds are valid in that parameter range. For other values of  $\mu$  we would, however, still need to calculate additional Lyapunov functions.

### 10.2 Error Bounds

We will now test our error bounds using the Lyapunov functions that are associated with  $P_1$ ,  $P_2$  and  $P_3$ , from section 10.1. The use of  $P_3$  is of particular interest since it is the only one that can be computed and used in an efficient manner. For the purpose of keeping things simple we will consider only the parameter value  $\mu = \bar{\mu}$  but figure 1 shows that the same bounds are also valid for a range of parameter values.

For our reduced model we will use six spatial basis elements ( $6 \times K$  total degrees of freedom). For the system's input we will use a constant value of one.

Table 1 gives the constants that are needed for our error bounds. The values of the constants  $\alpha_{Q_i}$  can also be seen in Figure 1. For comparison  $\alpha(\bar{\mu}) = -0.926712$ . For the  $L_2$  bound we can compare the values of  $C$  to 1.05768, which would be the value if we were using the original  $L_2$  bound.

Table 1. Constants needed for our error bounds

	$P_1$	$P_2$	$P_3$
$\alpha_{Q_i}$	1	0.614745	0.508327
$C$	0.980359	0.978926	0.978928

Figures 2 and 3 show the effectivity of our new error bounds over a range of time scales. Here effectivity is defined as the value of the error bound divided by the respective norm of the error. Since our bounds are rigorous, the effectivity can never fall below the ideal value of one.

The effectivity of the generalized energy bound is much better than that of the generalized  $L_2$  bound. That is generally also the case for the original energy and the  $L_2$  bounds. Nevertheless, the generalized  $L_2$  bound can be useful because it bounds the error at a specific time step.

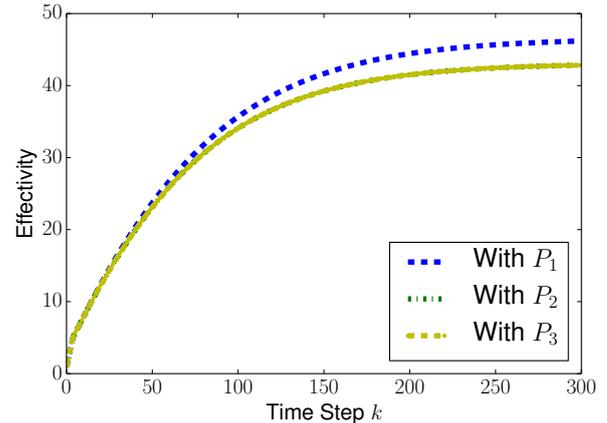


Fig. 3. Effectivity of the generalized  $L_2$  bounds

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