Hp-Adaptivity on Anisotropic Meshes for Hybridized Discontinuous Galerkin Scheme

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22nd AIAA Computational Fluid Dynamics Conference,
Dallas, Texas

June 23rd, 2015
Background and Motivation

Theme of Present work:

- Adjoint-based anisotropic (h and hp) adaptation

Context:

- Discontinuous Galerkin method for convection dominated flows
Anisotropic adaptation based on a continuous mesh model for triangular grids

Use two error estimators:

- Density of continuous mesh is determined by adjoint-based estimate
- For triangle of given area, anisotropy follows from analytic minimization of the interpolation error in the $L^q$-norm

The latter is based on recently proposed error model and estimates\(^1\)

For hp adaptation, choose the configuration - polynomial degree and the anisotropy - with smallest error estimate.

\(^1\)V. Dolejši., Applied Numerical Mathematics. 82, 2014.
Outline

1. Proposed Adaptation Method

2. Numerical Results
Mesh-Metric Duality

The discrete setting:

- Metric tensors encode information about mesh elements

- The triangular element is equilateral w.r.t the norm induced by the metric.

\[ \|e_k\|^2_M = e_k^T M e_k = C, \quad k = 1, 2, 3, \]

- The element is unit with respect to $M$

Continuous Mesh:

- Reconstruct continuous metric
- Unit Mesh generated using length constraint in Riemannian geometry.
Defining Size and Anisotropy

- Eigenvalue decomposition of the metric gives the size and shape of the triangle.

\[
\mathbf{M} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
\]

\[
\lambda_1 = \frac{1}{h_1^2}, \quad \lambda_2 = \frac{1}{h_2^2},
\]

- The area of the triangle is given for \( C = 3 \),

\[
l = \frac{3\sqrt{3}}{4} h_1 h_2
\]

- We define the aspect ratio of the mesh element as

\[
\beta = h_2 / h_1,
\]

- The triangle is described by \( \{l, \beta, \theta\} \).
Proposed Adaptation Method

Algorithm for h-Adaptation

1. Find the implied metric of the current mesh.

2. Determine new area, \( l \) of mesh elements using adjoint-based error estimate.
   - If the elemental error > the cut off error, then refine \( l = l_c/r_f \)
   - If the elemental error < the cut off error, then coarsen \( l = l_c \times c_f \)

3. Determine anisotropy \( \{\beta, \theta\} \) by minimization of error model in the \( L^q \) norm.

4. \( \{l, \beta, \theta\} \) is used to evaluate the desired metric on the current mesh and is fed to a metric-conforming mesh generator.

5. Do computation on new mesh

6. Done? Otherwise go to step 1
Define the error function $E$, of degree $p$, as

$$E_{\bar{x},p}(x) = \frac{1}{(p + 1)!} \sum_{l=0}^{p+1} \binom{p + 1}{l} \frac{\partial^{p+1} u(\bar{x})}{\partial x^l \partial y^{p+1-l}} (x - \bar{x})^l (y - \bar{y})^{p+1-l}$$
Define the error function $E_\bar{x},p(x)$, of degree $p$, as

$$E_\bar{x},p(x) = \frac{1}{(p+1)!} \sum_{l=0}^{p+1} \binom{p+1}{l} \frac{\partial^{p+1} u(\bar{x})}{\partial x^l \partial y^{p+1-l}}(x - \bar{x})^l(y - \bar{y})^{p+1-l}$$

The error model in radial coordinates $(x, y) = (|x| \cos \phi, |x| \sin \phi)$

$$E_\bar{x},p(x) = \frac{1}{(p+1)!} u^{(p+1, \phi)} |x - \bar{x}|^{p+1}$$ (1)

where the directional derivative of order $k$ is given by

$$u^{(k, \phi)} = \sum_{l=0}^{k} \binom{k}{l} \frac{\partial^k u}{\partial x^l \partial y^{k-l}} (\cos \phi)^l (\sin \phi)^{k-l}$$

Error model for interpolation of $u(x)$ is that of Taylor series $\pi_{\bar{x},p} u(x)$

$$u(x) - \pi_{\bar{x},p} u(x) \approx E_{\bar{x},p}(x)$$
A bound for the interpolation error function $E_{\bar{x},p}(x)$ can be written, in terms of 3 parameters $\{A_p, \rho_p, \phi_p\}$ as

$$|E_{\bar{x},p}(x)| \leq A_p \left((x - \bar{x})^T Q_{\phi_p} D_{\rho_p} Q_{\phi_p}^T (x - \bar{x})\right)^{\frac{p+1}{2}} \forall x \in \Omega$$

where,

$$Q_{\phi_p} = \begin{bmatrix} \cos \phi_p & -\sin \phi_p \\ \sin \phi_p & \cos \phi_p \end{bmatrix}, \quad D_{\rho_p} = \begin{bmatrix} 1 & 0 \\ 0 & \rho_p^{\frac{2}{p+1}} \end{bmatrix}$$

Note: $\{A_p, \rho_p\}$ modified from $\{\tilde{A}_p, \tilde{\rho}_p\}$, such that bound is safe (and sharp).
A bound for the interpolation error function $E_{\bar{x},p}(x)$ can be written, in terms of 3 parameters $\{\tilde{A}_p, \tilde{\rho}_p, \phi_p\}$ as

$$|E_{\bar{x},p}(x)| \lesssim \tilde{A}_p \left( (x - \bar{x})^T Q_{\phi_p} D\tilde{\rho}_p Q_{\phi_p}^T (x - \bar{x}) \right)^{\frac{p+1}{2}} \quad \forall x \in \Omega$$

where,

$$Q_{\phi_p} = \begin{bmatrix} \cos \phi_p & -\sin \phi_p \\ \sin \phi_p & \cos \phi_p \end{bmatrix}, \quad D_{\rho_p} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\rho}_p - \frac{2}{p+1} \end{bmatrix}$$
Proposed Adaptation Method

Interpolation error function and bound

A bound for the interpolation error function $E_{\bar{x},p}(x)$ can be written, in terms of 3 parameters $\{\tilde{A}_p, \tilde{\rho}_p, \phi_p\}$ as

$$|E_{\bar{x},p}(x)| \lesssim \tilde{A}_p \left( (x - \bar{x})^T Q_{\phi_p} D_{\tilde{\rho}_p} Q_{\phi_p}^T (x - \bar{x}) \right)^{\frac{p+1}{2}} \forall x \in \Omega$$

where,

$$Q_{\phi_p} = \begin{bmatrix} \cos \phi_p & -\sin \phi_p \\ \sin \phi_p & \cos \phi_p \end{bmatrix}, \quad D_{\rho_p} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\rho}_p - \frac{2}{p+1} \end{bmatrix}$$

and

$$\tilde{A}_p = \frac{1}{(p+1)!} |u^{(p+1,\phi_p)}|, \quad \tilde{\rho}_p = \frac{u^{(p+1,\phi_p)}}{u^{(p+1,\phi_p-\frac{\pi}{2})}}$$

$$\phi_p = arg\max_{\phi} |u^{(p+1,\phi)}|$$

Note: $\{A_p, \rho_p\}$ modified from $\{\tilde{A}_p, \tilde{\rho}_p\}$, such that bound is safe (and sharp)
For given density, we can analytically find the optimal anisotropy.

done by minimizing the bound in $L^q$-norm taken over metric ellipse.

The optimal anisotropy of the triangle is

$$\beta = \rho^{p+1}, \quad \theta = \phi - \pi/2,$$

The minimum bound is given as

$$\|E_{k,p}\|_{L^q(K)} \leq \omega$$

where

$$\omega = c_{p,q} A_p \rho^{0.5} \left( \frac{p+1}{2} + \frac{1}{q} \right)$$

and $c_{p,q}$ depends only on $p$ and $q$. 
Algorithm for Hp-Adaptation

1. Set the area $I$, of the triangle, $κ$ using the adjoint error estimate.

2. do a "dry run": Compute optimal triangle $p = p_κ - 1, p_κ, p_κ + 1$

3. Evaluate the error bound $ω$ using \{\(A_p, ρ_p, φ_p\)\} for the above three cases

4. Select, for each element $κ$, the $p$ and the corresponding metric \{\(I, β, θ\)\}, which gives the smallest error bound $ω$. 
Outline

1. Proposed Adaptation Method
2. Numerical Results
Solver details

- HDG scheme, with adjoint based estimator
- Damped Newton time integration
- GMRES, ILU(3) from PETSc as linear solver
- Netgen\(^2\) as (isotropic) mesh generator, Ngsolve as FE library
- BAMG\(^3\) as anisotropic mesh generator

\(^3\)F. Hecht, Technical report, Inria, France, 2006
Verification of the bound

The bound for the interpolation error is given as,

\[ |E_{\hat{x},p}(x)| \leq A_p \left( (x - \hat{x})^T Q_{\phi_p} D_{\rho_p} Q_{\phi_p}^T (x - \hat{x}) \right)^{\frac{p+1}{2}} \quad \forall x \in \Omega \]

Scalar boundary layer

\[ w(x, y) = \left( x + \frac{e^{x/\epsilon} - 1}{1 - e^{1/\epsilon}} \right) \cdot \left( y + \frac{e^{y/\epsilon} - 1}{1 - e^{1/\epsilon}} \right) \]

Figure: $\epsilon = 0.1$
Verification of the bound

Interpolation Error and Numerical Bound with $\{\tilde{A}_p, \tilde{\rho}_p\}$

Uniform Isotropic Refinement, $\epsilon = 0.1$
Verification of the bound

Interpolation Error and Numerical Bound with \( \{\tilde{A}_p, \tilde{\rho}_p\} \)

**Uniform Isotropic** Refinement, \( \epsilon = 0.1 \)

![Graph showing the relationship between number of elements and error bound.](image-url)
Verification of the bound

Interpolation Error and Numerical Bound with $\{\tilde{A}_p, \tilde{\rho}_p\}$

*Uniform Isotropic* Refinement, $\epsilon = 0.1$

![Graph showing $L^2$ norm against number of elements](image-url)
Verification of the bound

Interpolation Error and Numerical Bound with \( \{\tilde{A}_p, \tilde{\rho}_p\} \)

**Uniform Isotropic** Refinement, \( \epsilon = 0.1 \)

\[
\begin{array}{c}
\times 10^{-8} \\
10^2 \\
10^3 \\
10^4 \\
\end{array}
\]

- - - 4 error
- - 4 bound

Number of elements

\( L^2 \) norm
Verification of the bound

Interpolation Error and Numerical Bound with \( \{ \tilde{A}_p, \tilde{\rho}_p \} \)

Uniform Isotropic Refinement, \( \epsilon = 0.1 \)

![Graph showing the relationship between the number of elements and L2 norm for different polynomial degrees.](image)
Numerical Results

Verification of the bound

Interpolation Error, Numerical Bound with \( \{\tilde{A}_p, \tilde{\rho}_p\} \) and Modified Bound with \( \{A_p, \rho_p\} \)

Adaptive Anisotropic Refinement, \( \epsilon = 0.1 \)

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Graph showing the relationship between the number of elements and the \( L^2 \) norm for different error bounds.
Verification of the bound

Interpolation Error, Numerical Bound with $\{\tilde{A}_p, \tilde{\rho}_p\}$ and Modified Bound with $\{A_p, \rho_p\}$

Adaptive Anisotropic Refinement, $\epsilon = 0.1$

- - - $p = 3$ error
- - - $p = 3$ bound
- - - $p = 3$ modified bound

$L^2$ norm

Number of elements
Verification of the bound

Interpolation Error, Numerical Bound with \( \{ \tilde{A}_p, \tilde{\rho}_p \} \) and Modified Bound with \( \{ A_p, \rho_p \} \)

Adaptive Anisotropic Refinement, \( \epsilon = 0.1 \)
Verification of the bound

Interpolation Error, Numerical Bound with $\{\tilde{A}_p, \tilde{\rho}_p\}$ and Modified Numerical Bound with $\{A_p, \rho_p\}$

Adaptive Anisotropic Refinement, $\epsilon = 0.1$
Verification of the bound

Element-wise distribution of interpolation error and numerical bound (sorted)

Adapted Anisotropic Mesh, 1581 elements, $\epsilon = 0.1$, $p = 3$

Error vs. Numerical Bound with $\{A_p, \tilde{\rho}_p\}$

Error vs. Modified numerical bound with $\{A_p, \rho_p\}$
Verification of the bound

Element-wise distribution of interpolation error and numerical bound (sorted)
Adapted Anisotropic Mesh, 1450 elements, $\epsilon = 0.01$, $p = 3$

Error vs. Numerical Bound with \{\tilde{A}_p, \tilde{\rho}_p\}  

Error vs. Modified numerical bound with \{A_p, \rho_p\}
Numerical Results

Verification of the bound

Element-wise distribution of interpolation error and numerical bound (sorted)
Adapted Anisotropic Mesh, 1402 elements, $\epsilon = 0.005$, $p = 3$

![Graph 1: Error vs. Numerical Bound with $\{\tilde{A}_p, \tilde{\rho}_p\}$](image1)

![Graph 2: Error vs. Modified numerical bound with $\{A_p, \rho_p\}$](image2)
Scalar convection-diffusion

2D viscous Burger’s equation with source

\[
\nabla \cdot (\mathbf{w}, \mathbf{w}) - \epsilon \Delta \mathbf{w} = s \\
w(x, y) = 0
\]

\[(x, y) \in \Omega = [0, 1]^2 \quad (x, y) \in \partial \Omega\]

We take the solution as

\[
w(x, y) = \left(x + \frac{e^{x/\epsilon} - 1}{1 - e^{1/\epsilon}}\right) \cdot \left(y + \frac{e^{y/\epsilon} - 1}{1 - e^{1/\epsilon}}\right),
\]

The target functional of interest is the weighted total boundary flux i.e.

\[
J = \int_{\partial \Omega} \psi (\mathbf{w} - \epsilon n \cdot \mathbf{q}) d\sigma
\]

where the weighting function, \(\psi = \cos(2\pi x) \cos(2\pi y)\).
Numerical Results

Scalar convection-diffusion

**h-Adaptation, $p = 2, \epsilon = 0.005$**

Initial and adapted mesh

Initial mesh - 512 elements

h-adapted mesh, 2nd adaptation step - 1417 elements
Numerical Results

Scalar convection-diffusion

h-Adaptation, $p = 2$, $\epsilon = 0.005$

Solution on initial and h-adapted mesh

Initial mesh - 512 elements

Adapted mesh - 1417 elements
Numerical Results

Scalar convection-diffusion

Comparison of isotropic and anisotropic adaptations.

Error Vs. Degrees of freedom
Scalar convection-diffusion

Comparison of isotropic and anisotropic adaptations.

![Graph showing error vs. degrees of freedom for isotropic and anisotropic adaptations.](image-url)
Numerical Results

Scalar convection-diffusion

Comparison of h- and hp-Adaptation, $\epsilon = 0.005$
Solution at same error level of $\approx 10^{-9}$

H-adapted mesh - 1417 elements, $p = 2$  
Hp-adapted mesh - 554 elements, $p = 2, \ldots, 6$
Numerical Results

Flow over NACA0012 airfoil

Target functional - Drag coefficient

Flow cases:

- Subsonic viscous flow, $M_\infty = 0.5, \alpha = 1.0^\circ, \text{Re} = 5000$
- Subsonic inviscid flow, $M_\infty = 0.5, \alpha = 2^\circ$
- Transonic inviscid flow, $M_\infty = 0.8, \alpha = 1.25^\circ$
- Supersonic inviscid flow, $M_\infty = 1.5, \alpha = 0^\circ$

Initial Mesh - 2155 elements, Far field at a radius of 1000 chords.
Subsonic viscous flow over NACA0012

**h-Adaptation, \( p = 2 \)**
Adapted mesh and Mach number

Adapted mesh - 11945 elements

Mach number
Numerical Results

Subsonic viscous flow over NACA0012

*h-Adaptation*, $p = 2$
Capture of flow recirculation

Adapted mesh

Streamline
Subsonic viscous flow over NACA0012

Comparison of error convergence for isotropic and anisotropic adaptations.
For hp-adaptation, $p = 1, \ldots, 6$

Error in drag coefficient Vs Degrees of freedom
Numerical Results

Subsonic viscous flow over NACA0012

hp-Adaptation
Capture of flow recirculation

Adapted mesh - 5361 elements

Streamline
Subsonic viscous flow over NACA0012

Comparison of error convergence for isotropic and anisotropic adaptations. For hp-adaptation, $p = 1, \ldots, 6$

Error in drag coefficient Vs Degrees of freedom
Subsonic viscous flow over NACA0012

Comparison of error convergence for isotropic and anisotropic adaptations. For hp-adaptation, $p = 1, ..., 6$

Error in drag coefficient Vs Degrees of freedom
Inviscid subsonic flow over NACA0012

**Hp-Adaptation**, $p = 1, \ldots, 6$
Adapted mesh - 1674 elements
Inviscid subsonic flow over NACA0012

Comparison of Anisotropic and Isotropic Hp-adaptation

Polynomial map

Anisotropic adaptation, using Interpolation error

Isotropic adaptation, using Jump indicator
Inviscid subsonic flow over NACA0012

Comparison of error convergence for isotropic and anisotropic adaptations. For hp-adaptation, $p = 1, \ldots, 6$

Error in drag coefficient Vs Degrees of freedom
Numerical Results

Inviscid transonic flow over NACA0012

**H-Adaptation, \( p = 2 \)**
Adapted mesh and Mach number

Adapted mesh - 4394 elements  
Mach number
Inviscid transonic flow over NACA0012

Comparison of error convergence for isotropic and anisotropic adaptations.

Error in drag coefficient Vs Degrees of freedom
Numerical Results

Inviscid transonic flow over NACA0012

Hp-Adaptation, $p = 1, 2, ..., 6$
Adapted mesh, 4258 elements
Mach number and Polynomial map
Inviscid transonic flow over NACA0012

Comparison of error convergence for isotropic and anisotropic adaptations. For hp-adaptation, $p = 1, \ldots, 6$

![Graph showing error in drag coefficient vs. degrees of freedom](graph.png)

Error in drag coefficient Vs Degrees of freedom
Inviscid transonic flow over NACA0012

Comparison of error convergence for isotropic and anisotropic adaptations. For hp-adaptation, $p = 1, \ldots, 6$

![Graph showing error in drag coefficient vs degrees of freedom.](image)

Error in drag coefficient Vs Degrees of freedom
**Numerical Results**

**Inviscid supersonic flow over NACA0012**

**H-Adaptation, \( p = 2 \)**

Adapted mesh and Mach number

Adapted mesh - 7380 elements

Mach number
Inviscid supersonic flow over NACA0012

**H-Adaptation, \( p = 2 \)**
Adapted mesh near bow-shock

![Adapted mesh](image1)

![Adapted mesh](image2)
Inviscid supersonic flow over NACA0012

Hp-Adaptation, $p = 1, ..., 6$
Mach number and Polynomial map (8267 elements)
Inviscid supersonic flow over NACA0012

Comparison of error convergence for isotropic and anisotropic adaptations. For hp-adaptation, \( p = 1, \ldots, 6 \)

Error in drag coefficient Vs Degrees of freedom
Conclusions

- Interpolation bound and verification
- Anisotropic refinement - superior to isotropic refinement
- Hp-refinement - superior to pure h-refinement

Future work:
- Extension to 3D
- Global optimization
Financial support from the Deutsche Forschungsgemeinschaft (German Research Association) through grant GSC 111 is gratefully acknowledged.