A stable high-order Spectral Difference method for hyperbolic conservation laws on triangular elements

Aravind Balan\textsuperscript{a,}\textsuperscript{*}, Georg May\textsuperscript{a}, Joachim Schöberl\textsuperscript{b}

\textsuperscript{a}AICES Graduate School, RWTH Aachen University, Schinkelstr. 2, 52062 Aachen, Germany
\textsuperscript{b}Institute for Analysis and Scientific Computing, TU Vienna, Wiedner Hauptstr. 8-10/101, 1040 Vienna, Austria

\begin{abstract}
Numerical schemes using piecewise polynomial approximation are very popular for high order discretization of conservation laws. While the most widely used numerical scheme under this paradigm appears to be the Discontinuous Galerkin method, the Spectral Difference scheme has often been found attractive as well, because of its simplicity of formulation and implementation. However, recently it has been shown that the scheme is not linearly stable on triangles. In this paper we present an alternate formulation of the scheme, featuring a new flux interpolation technique using Raviart–Thomas spaces, which proves stable under a similar linear analysis in which the standard scheme failed. We demonstrate viability of the concept by showing linear stability both in the semi-discrete sense and for time stepping schemes of the SSP Runge–Kutta type. Furthermore, we present convergence studies, as well as case studies in compressible flow simulation using the Euler equations.
\end{abstract}

\section{Introduction}

For some time now a considerable amount of research activity has been devoted to high-order methods for the simulation of compressible fluid flow. In particular schemes based on piecewise continuous polynomial approximation, such as Discontinuous Galerkin (DG) methods\cite{1–5} became increasingly popular. Even for moderate levels of accuracy, high-order methods are potentially more efficient than low-order methods in terms of total number of degrees of freedom\cite{6}. However, for such problems, low order methods are still widely used as they have proved to be reliable and stable. For high-order methods to be viable in this context, they should be stable, robust and easy to implement. It is in this context that the SD scheme has been proposed as a collocation method derived from the strong form of the governing conservation laws\cite{7–9}. Recently, the connection between general nodal formulations, including the Spectral Difference (SD) scheme, and the weak form of the governing equations has been highlighted by several authors, using unifying formulations such as Huynh’s flux reconstruction schemes\cite{10}, energy-stable variants thereof, proposed by Vincent et al.\cite{11}, or Wang’s lifting collocation penalty schemes\cite{12}. By using the ‘quadrature-free’ paradigm\cite{13}, equivalence between Spectral Difference and nodal Discontinuous Galerkin methods can be established in particular for nonlinear conservation laws\cite{14}. The Spectral Difference scheme seems to be particularly attractive as there are no surface or volume integrals to be computed.

The SD scheme has been found to be stable for one dimensional linear advection problems by Jameson\cite{15} for all orders of accuracy in an energy norm of Sobolev type. But linear stability analysis performed by den Abeele et al.\cite{16} shows that the
scheme in its standard formulation is unstable for order of accuracy greater than two on triangular meshes. The new variant of the Spectral Difference scheme (SD-RT), which was proposed by May and Schöberl [17], is formulated by incorporating a new interpolation technique into the standard Spectral Difference scheme. In this scheme, the flux function of the conservation law is approximated by projecting it onto the Raviart–Thomas (RT) space. For triangular meshes, the scheme proved stable under linear stability analysis for periodic problems for which the standard Spectral Difference scheme was found unstable [17]. Further, the RT space demands fewer degrees of freedom for the same order of accuracy than the standard scheme, which means a reduction in computational cost.

In the present paper we build on the stability analysis in [17], and construct stable RT-based nodal elements for flux interpolation having divergence in the space of linear, quadratic, and cubic polynomials. First, in Sections 2 and 3, respectively, we recall the formulation of the standard Spectral Difference scheme, followed by the derivation of the Raviart–Thomas based scheme (SD-RT). In Section 4 we consider linear stability analysis as a design tool to produce stable elements for the SD-RT scheme. We consider both the semi-discrete case, as well as the fully discrete case using time stepping schemes of the SSP Runge–Kutta type [18]. Finally, we demonstrate viability of the SD-RT scheme with convergence studies using the scalar advection equation and the Euler equations in Section 5, and case studies in inviscid external aerodynamics in Section 6.

## 2. The standard Spectral Difference scheme for triangles

We present here the derivation of the standard SD scheme from the strong form of the governing equations as given in the original formulation of the scheme [7]. Consider the scalar hyperbolic conservation equation

\[ \frac{\partial u(x,t)}{\partial t} + \nabla \cdot \vec{f}(u) = 0 \]  

on some domain \((x,t) \in \Omega \times \mathbb{R}^+\), where \(\Omega \subset \mathbb{R}^2\) in the present case. Note that we reserve vector notation for the flux function, as this will prove useful for highlighting the differences in the flux interpolation between the standard SD scheme, derived in this section, and the new scheme. Consider a triangulation of the domain, \(T_h = \{T^i, i = 1, \ldots, N_T\}\) which decomposes the domain \(\Omega\) into \(N_T\) cells, where each cell is denoted by the index \(i\) as a bracketed superscript. However, the formulations given below hold for each cell and the superscript is omitted to avoid confusion and is shown only in the final form of the equation. For convenience, the SD scheme is formulated for a reference triangle \(\bar{T}\). Consider an invertible mapping from the reference domain \((\xi)\) to the physical domain \((x)\) for each cell, defined by \(\Phi: \xi \mapsto x\) with Jacobian \(J = \partial \xi / \partial x\), such that \(\bar{T} = \Phi^{-1}(T)\). Two sets of points are defined in the reference domain. The first set, \(\bar{\xi}_j, j = 1, \ldots, N_m\), is for the solution collocation and the second one, \(\bar{\xi}_k, k = 1, \ldots, N_{m+1}\), is for the flux collocation. In the reference domain, if \(\Phi\) is independent of time, the above hyperbolic equation is of the form

\[ \frac{\partial u(\xi,t)}{\partial t} + \frac{1}{|J|} \nabla \cdot \left( |J| \bar{f}(u) \right) = 0. \]  

The solution in the reference domain is approximated to \(u_h\) by

\[ u_h(\xi) = \sum_{j=1}^{N_m} u_j \bar{l}_j(\xi), \]  

where \(\bar{l}_j(\xi), j = 1, \ldots, N_m\) are the Lagrangian interpolation functions, defined by the solution collocation nodes with the property \(u_j(\bar{\xi}_j) = \delta_{jk}\), and hence the coefficients are given as \(u_j = u_h(\bar{\xi}_j)\). These Lagrangian functions form a basis for the space of polynomials of degree \(m\), denoted as \(P_m\). In 2D, the number of degrees of freedom required to represent a function in the space \(P_m\) is given as

\[ N_m = \frac{(m+1)(m+2)}{2}. \]

The flux function in the reference domain, which is defined as \(|J|^{-1} \bar{f}\), is approximated by \(\bar{f}_h\), by projecting it component-wise into a finite dimensional polynomial space of degree \(m + 1\) as

\[ \bar{f}_h(\xi) = \sum_{k=1}^{N_{m+1}} \bar{f}_k \bar{l}_k(\xi), \]  

where \(\bar{l}_k(\xi), k = 1, \ldots, N_{m+1}\) are the Lagrangian interpolation functions, defined by the flux collocation nodes with the property \(\bar{l}_k(\bar{\xi}_k) = \delta_{jk}\). The coefficients of the interpolation are defined as

\[ \bar{f}_k = \begin{cases} |J|^{-1} \bar{f}(\bar{\xi}_k), & \bar{\xi}_k \in \bar{T}, \\ |J|^{-1} \bar{f}_{num}, & \bar{\xi}_k \in \partial \bar{T}, \end{cases} \]

where we require for the normal flux on edges

\[ |J|^{-1} \bar{f}_{num} \cdot n^i = g. \]
Here $n^i$ is the outward normal on $\partial \bar{T}$, and $g$ is a consistent and conservative numerical flux function, taking input values from both triangles adjacent to an interior edge, or from a suitable boundary function for \( \zeta \in \partial \Omega \). The component of \( \bar{f}_{num} \) tangential to edges may be taken from the interior trace of the element.

Note that the flux function is approximated componentwise to one degree higher than the solution function. This is to ensure that the divergence of the flux will be a polynomial of degree $m$. The number of degrees of freedom for the flux approximation (including both $x$ and $y$ components) is thus $2N_{m+1}$. The divergence of the flux function in the finite dimensional polynomial space is given as

$$
\left( \nabla^i \cdot \bar{f}_h \right)(\zeta) = \sum_{k=1}^{N_{m+1}} \left( \nabla^i \tilde{f}_h \right)(\zeta) \cdot \tilde{f}_k.
$$

(8)

The Spectral Difference scheme can now be written for each degree of freedom of the solution function in each cell $i$ as

$$
\frac{d u^{(i)}}{dt} + \frac{1}{|\bar{f}|} \sum_{k=1}^{N_{m+1}} \left( \nabla^i \tilde{f}_h \right)(\zeta_j) \cdot \tilde{f}_k = 0, \quad j = 1, \ldots, N_m, \quad i = 1, \ldots, N_T.
$$

(9)

The above Spectral Difference scheme was found unconditionally linearly unstable for triangles for $m > 1$ [16].

3. The Spectral Difference scheme for triangles using Raviart–Thomas elements (SD-RT)

The numerical solution is approximated as in Eq. (3) for the standard Spectral Difference scheme. For the flux function, vector valued interpolation is used where the interpolation functions are vectors in the Raviart–Thomas (RT) space. For a degree $m$, the RT space is defined as

$$RT_m = \left( P_m \right)^2 + (x, y)^T P_m.$$  

(10)

Since $RT_m$ is the smallest space having divergence in $P_m$ [19], it reduces the number of degrees of freedom needed for flux interpolation and hence the computational cost is reduced compared to that for the traditional Spectral Difference scheme. In 2D, the number of degrees of freedom to represent a vector-valued function in the RT space is given by

$$N_{m}^{RT} = (m + 1)(m + 3).$$

(11)

The flux function in the reference domain, which is defined as \( J J^{-1} \bar{f} \), is approximated to $\bar{f}_h$ in the RT space as

$$\bar{f}_h(\zeta) = \sum_{k=1}^{N_{m}^{RT}} f_k \tilde{\psi}_k(\zeta),$$

(12)

where $\tilde{\psi}_k$, $k = 1, \ldots, N_{m}^{RT}$, are interpolation functions which form a basis in the Raviart–Thomas space of degree $m$. Compared to the standard SD scheme, the number of degrees of freedom for the flux function in SD-RT scheme is reduced by

$$\Delta_{DOF} = 2N_{m+1} - N_{m}^{RT} = m + 3.$$  

(13)

Note that the basis functions are vectors and the coefficients are scalars unlike the standard Spectral Difference where the basis is scalar and the coefficients are vectors. Further, these interpolation functions have the property $\tilde{\psi}_k(\zeta_k) \cdot s_k = \delta_{jk}$, where $\tilde{\psi}_k$, $k = 1, \ldots, N_{m}^{RT}$, are the flux collocation points and $s_k$ are the unit vectors defined at those points. For a degree $m$, according to standard theory [19], there should be $m + 1$ points on each edge of the triangle and the remaining $N_{m}^{RT} - 3(m + 1)$ in the interior. The degrees of freedom for the flux interpolation are given as

$$f_k = \begin{cases} 
& \left( J J^{-1} \bar{f} \tilde{\psi}_k \right) \cdot s_k, \quad \tilde{\psi}_k \in \bar{T}, \\
& g, \quad \tilde{\psi}_k \in \partial \bar{T},
\end{cases}$$

(14)

where $g$ is a numerical flux, just as in the standard formulation. Note, however, that the numerical flux is directly used as the degree of freedom on edges, unlike the standard Spectral Difference where the numerical flux is used to replace the normal component of the analytical flux and is then projected into orthogonal directions to obtain the degrees of freedom [7]. The divergence of the flux function at the solution nodes is given as

$$\left( \nabla^i \cdot \bar{f}_h \right)(\zeta_j) = \sum_{k=1}^{N_{m}^{RT}} f_k \left( \nabla^i \cdot \tilde{\psi}_k \right)(\zeta_j), \quad j = 1, \ldots, N_m.$$

(15)

The values of $\left( \nabla^i \cdot \tilde{\psi}_k \right)(\zeta_j)$ form a differentiation matrix. The SD-RT scheme can now be written for each degree of freedom of the solution function in each cell $i$ as

$$\frac{d u^{(i)}}{dt} + \frac{1}{|\bar{f}|} \sum_{k=1}^{N_{m}^{RT}} f_k \left( \nabla^i \cdot \tilde{\psi}_k \right)(\zeta_j) = 0, \quad j = 1, \ldots, N_m, \quad i = 1, \ldots, N_T.$$  

(16)
4. Linear stability analysis

The SD-RT scheme has been subject to a linear stability analysis by May and Schöberl [17], which is extended here. The linear stability of the SD-RT scheme is independent of the position of solution nodes but depends on the position of flux nodes like for the standard SD scheme. The analysis is done on SD-RT₁, SD-RT₂ and SD-RT₃ schemes to find the position of flux nodes for which the schemes are linearly stable. We consider the two-dimensional linear advection

\[
\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f}(u) = 0,
\]

with \( \mathbf{f}(u) = (c_1 u, c_2 u) \), where \( c_1 = |c|\cos \theta \), \( c_2 = |c|\sin \theta \) for the advection angle \( \theta \in [0, \pi] \) and \( |c| \) being the magnitude of advection velocity. A two-dimensional uniform skewed mesh with skew angle \( \mu \in (0, \pi/2) \), as shown in Fig. 1, is created and each mesh element, identified by \((i,j)\), is subdivided into two triangular elements. The SD-RT scheme is then formulated for these elements, using upwind fluxes on the edges, to yield

\[
\Delta t \frac{dU^{(i,j)}}{dt} = \begin{cases} -v(AU^{(i,j)} + BU^{(i-1,j)} + CU^{(i,j-1)}), & \forall \theta \leq \mu \\ -v(DU^{(i,j)} + EU^{(i+1,j)} + FU^{(i,j-1)}), & \forall \theta > \mu \end{cases}
\]

where \( U = (u_1, u_2, \ldots, u_{2n_\text{m}})^T \) comprises all solution degrees of freedom in the Cartesian mesh element \((i,j)\). The CFL number is given as \( v = \frac{\Delta t}{\Delta x} \), where \( \Delta x \) is the Cartesian mesh edge length. \( A, B, \ C, \ D, \ E \) and \( F \) are matrices of dimension \( 2N_\text{m} \times 2N_\text{m} \).

4.1. Linear stability analysis for the spatial discretization

The analysis considered here is based on linear stability analysis for periodic problems [20]. The solution is decomposed into different frequency modes by Fourier transformation and the behavior of these individual modes is analyzed. If \( U e^{(k_x x + k_y y)} \) is one particular mode with \( k_x, k_y \) being the wave numbers in \( x \) and \( y \) directions, respectively, then the SD-RT discretization for this mode will take the form

\[
\Delta t \frac{d\tilde{U}^{(0,j)}}{dt} = \nu Z \tilde{U},
\]

where

\[
Z = \begin{cases} -(A + Be^{-i\sigma} + Ce^{-ik}), & \forall \theta \leq \mu \\ -(D + Ee^{-i\sigma} + Fe^{-ik}), & \forall \theta > \mu \end{cases}
\]

is the Fourier symbol of the spatial discretization, and \((\sigma, \kappa) = (k_x, k_y, \mu, h)\) defines the grid frequency. The numerical stability of Eq. (20) depends on the eigensystem of matrix \( Z \). The eigenvalues \( \lambda \) of \( Z \) are evaluated numerically at discrete grid frequencies \((\sigma, \kappa) \in [0, 2\pi]^2\). For spatially stable discretization, all the eigenvalues of \( Z \) should have non-positive real part. As mentioned in Section 3, the SD-RT₃ scheme has \( k + 1 \) flux nodes on each edge of the triangle, while remaining flux degrees of freedom are located in the interior. We place edge nodes at Legendre–Gauss quadrature points [21]. It has been found numerically that the placement of flux nodes on the edges does not influence the spectrum and hence the linear stability
properties for SD-RT1, SD-RT2 and SD-RT3 schemes are not affected, whereas the flux node placement in the interior of the triangle has considerable effects on the spectrum and hence on the stability. As mentioned in Section 3, degrees of freedom for the flux function are determined by using both flux nodes and a unit vector at those flux nodes. Here the interior flux points are chosen such that one point with two mutually orthogonal unit vectors form two degrees of freedom as shown in Fig. 3. For SD-RT1, the two interior degrees of freedom are put at the centroid, for symmetry reasons, and this configuration was found to be stable.

In the case of SD-RT2, there are six interior degrees of freedom (three points each with two unit vectors). The position of these are varied linearly by a scaling parameter \( \alpha \in (0, 1) \), such that, if \( \xi_i \), \( i = 1, 2, 3 \) are the three vertices of the reference triangle, and \( \xi_c \) is the centroid, the interior flux points \( \zeta_i \) are given by

\[
\zeta_i = \xi_i + \alpha(\xi_i - \xi_c), \quad i = 1, 2, 3.
\]

The maximum eigenvalue of \( Z \) is found numerically, considering sufficiently many grid frequencies \((\sigma, \kappa)\), advection angles \( \theta \) and skew angles \( \lambda \), for different values of \( \alpha \). Fig. 2(a) shows the maximum value of the real part of the eigenvalues \((\lambda)\) of \( Z \) for different values of the scaling parameter. For \( \alpha < 0.5 \), the discretization is found to be unstable as the real part of the eigenvalues is positive. There is another region of instability, where the eigenvalues have very low magnitude positive real part. This is shown in Fig. 2(b) which has a higher resolution. From the numerical analysis it is found that the discretization is stable for all considered advection angles and skew angles if \( 0.5 < \alpha < 0.521 \).

The maximum spectral radius of \( Z \) is also calculated and is found to increase with the scaling parameter. Considering the fact that the smallest spectral radius is desirable to get the maximum stable CFL number for time integration, and the above observations on the region of stability, \( \alpha = 0.5 \) seems to be the optimal choice for flux node placement for SD-RT2 discretization. Interestingly, this optimal choice corresponds to using high-order (optimal) quadrature points in the interior of triangles, supporting exact integration of quadratic polynomials [22]. Similar linear stability analysis showed that the traditional Spectral Difference scheme of the same order failed in having a stable flux-point distribution [16].

For SD-RT3, different sets of interior flux points obtained by varying \( \alpha \) does not give any stable discretization. However, the stable SD-RT3 discretization with quadrature points motivated us to try the high-order quadrature rules for SD-RT3 discretization as well. Interestingly, it was found that using the points of a six-point quadrature rule [17], which is exact for polynomials of total degree 4, resulted in stable configurations for all advection and skew angles.

Fig. 3 shows the flux node distribution used for SD-RT1, SD-RT2 and SD-RT3 schemes we have used in the present paper. On the edges, the nodes are placed at the Gauss–Legendre quadrature points [21]. Interior flux points for SD-RT1 are in the centroid, those for SD-RT2 element correspond to \( \alpha = 0.5 \) and those for SD-RT3 scheme correspond to the points obtained using the six-point quadrature rule. Fig. 4 shows the spectrum of the Fourier symbol \((Z)\) for the above mentioned stable choice of flux nodes for SD-RT\( m \) schemes for the advection angle \( \theta = 45^\circ \) and skew angle \( \mu = 60^\circ \).

4.2. Linear stability analysis for the full discretization

The stable flux points shown in Fig. 3 are used here for the linear stability analysis of the full discretization. We consider explicit Runge–Kutta schemes for the time derivative term in Eq. (20), where the solution at \((n + 1)\)th iteration, \( U^{n+1} \), is obtained from \( U^n \) as

\[
\begin{align*}
\text{(a) Maximum of the real part of eigenvalues vs. scaling parameter } \alpha. \\
\text{(b) Maximum of the real part of eigenvalues vs. scaling parameter } \alpha \text{ (high resolution).}
\end{align*}
\]

Fig. 2. Influence of the scaling parameter \( \alpha \) on the spectrum of the Fourier symbol \( Z \) for the SD-RT\( 2 \) scheme.
Fig. 3. RT₁ (top left), RT₂ (top right) and RT₃ (bottom) elements.

Fig. 4. Spectrum of the Fourier symbol (Z) for the stable choice of flux nodes for SD-RT₁ (top left), SD-RT₂ (top right) and SD-RT₃ (bottom) schemes for the advection angle $\vartheta = \frac{45}{176}$, skew angle $\mu = \frac{60}{176}$, and discrete values of $(\sigma, \lambda) \in [0, 2\pi]^2$. 
We consider a recently proposed 5-stage SSP Runge–Kutta scheme\cite{24}, with coefficients given as
\begin{equation}
W^{(0)} = \bar{U}^n,
W^{(k)} = \sum_{l=0}^{k-1} \alpha_{kl} W^{(l)} + v \beta_{kl} Z W^{(l)} \quad k = 1, \ldots, p,
\end{equation}
where \( p \) is the number of intermediate stages and \( \Delta t \) is the time step. Note that the bracketed superscript here stands for different stages and not for different cells. The amplification factor \( G \), such that \( \bar{U}^{n+1} = G \bar{U}^n \), depends on the advection angle \( \theta \), the skew angle \( \mu \), the grid frequencies \( (\sigma, \kappa) \) and the CFL number \( \nu \). If \( G^{(k)} \) is the amplification matrix in the \( k \)th intermediate step, then one obtains
\begin{equation}
G^{(0)} = I, \quad G^{(k)} = \sum_{l=0}^{k-1} (\alpha_{kl} I + v \beta_{kl} Z) G^{(l)} \quad k = 1, \ldots, p.
\end{equation}

\( G \equiv \tilde{G}^{(p)} \) is the amplification matrix for the update from \( \tilde{U}^n \) to \( \tilde{U}^{n+1} \). For a stable discretization, it is sufficient to have \( \rho(G) \leq 1 \), where \( \rho \) is the spectral radius of \( G \). In other words, for a stable time and space discretization, all the eigenvalues of \( G \) must lie inside the unit circle in the complex plane. Here \( \rho \) is taken as the maximum eigenvalue found from sufficiently many grid frequencies \( (\sigma, \kappa) \). The maximum allowable CFL numbers, such that \( \rho(G) \leq 1 \), for different time discretization methods are found numerically for different values of advection angles \( \theta \) and skew angles \( \mu \). We test TVD Runge–Kutta schemes proposed by Gottlieb and Shu\cite{23}. For a 3 stage \(( p = 3 \) 3rd order scheme (Shu-RK3), the coefficients are given in matrix form as
\begin{equation}
\alpha = \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{2}{3} \\
\frac{1}{3} & 0 & 0
\end{bmatrix}, \quad \beta = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\end{equation}

For a 2 stage, 2nd order scheme (Shu-RK2), the coefficients are given as
\begin{equation}
\alpha = \begin{bmatrix}
1 \\
\frac{1}{2} \\
\frac{1}{2}
\end{bmatrix}, \quad \beta = \begin{bmatrix}
1 \\
0 \\
\frac{1}{2}
\end{bmatrix}.
\end{equation}

Following the successful application of TVD Runge–Kutta schemes for hyperbolic partial differential equations, a new general class of Strong Stability Preserving (SSP) schemes, which includes TVD-RK schemes, was developed by Gottlieb and Shu\cite{18}. We consider a recently proposed 5-stage SSP Runge–Kutta scheme\cite{24}, with coefficients given as
\begin{equation}
\alpha = \begin{bmatrix}
1 & 0.444370494067 & 0.555629505933 \\
0.620101851385 & 0 & 0.379898148615 \\
0.178079954108 & 0 & 0.821920045892 \\
0.006833258840 & 0 & 0.517231672090 \end{bmatrix},
\end{equation}
\begin{equation}
\beta = \begin{bmatrix}
0.391752227004 & 0.368410592630 & 0.352 & 0.9298 & 0.267 \\
0.215 & 0.225 & 0.215 & 0.186 & 0.168 \\
0.140 & 0.141 & 0.140 & 0.117 & 0.109
\end{bmatrix}.
\end{equation}

For both SD-RT2 and SD-RT3 schemes with Shu-RK2 time discretization, numerical evidence indicates that \( \max_{\sigma, \kappa} \rho(G) \) is always strictly greater than one for all CFL numbers until machine accuracy is approached. This means no stable CFL numbers were found up to the point where further analysis becomes unreliable due to the finite machine precision. Due to the observed behavior we view the Shu-RK2 scheme as impractical, and therefore excluded it from further analysis. Both the Shu-RK3 scheme and the 5-stage 4th order SSP scheme are stable under a CFL condition for all advection and skew angles. Tables 1 and 2 show the maximum stable CFL numbers for a few advection angles and skew angle \( \mu = \pi/4 \).

<table>
<thead>
<tr>
<th>SD-RT_m</th>
<th>0</th>
<th>\pi/8</th>
<th>\pi/4</th>
<th>3\pi/8</th>
<th>\pi/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>SD-RT1</td>
<td>0.352</td>
<td>0.367</td>
<td>0.352</td>
<td>0.298</td>
<td>0.267</td>
</tr>
<tr>
<td>SD-RT2</td>
<td>0.215</td>
<td>0.225</td>
<td>0.215</td>
<td>0.186</td>
<td>0.168</td>
</tr>
<tr>
<td>SD-RT3</td>
<td>0.140</td>
<td>0.141</td>
<td>0.140</td>
<td>0.117</td>
<td>0.109</td>
</tr>
</tbody>
</table>

Table 1

The maximum stable CFL numbers when Shu-RK3 time discretization is used for SD-RT_m \((m = 1, 2, 3)\) schemes. The skew angle in these cases is 45°.
Table 2
The maximum stable CFL numbers when 4th order SSP time discretization is used for SD-RT_m (m = 1, 2 and 3) schemes. The skew angle in all these cases is 45°.

<table>
<thead>
<tr>
<th>SD-RT_m</th>
<th>( \theta = 0 )</th>
<th>( \theta = \pi/8 )</th>
<th>( \theta = \pi/4 )</th>
<th>( \theta = 3\pi/8 )</th>
<th>( \theta = \pi/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SD-RT_1</td>
<td>0.564</td>
<td>0.575</td>
<td>0.564</td>
<td>0.473</td>
<td>0.427</td>
</tr>
<tr>
<td>SD-RT_2</td>
<td>0.337</td>
<td>0.368</td>
<td>0.337</td>
<td>0.299</td>
<td>0.267</td>
</tr>
<tr>
<td>SD-RT_3</td>
<td>0.223</td>
<td>0.242</td>
<td>0.223</td>
<td>0.200</td>
<td>0.179</td>
</tr>
</tbody>
</table>

The minimum of the maximum stable CFL numbers taken over all advection angles for Shu-RK3 and 4th order SSP for SD-RT_2 discretization are plotted against the skew angles in Fig. 5(a). From the figure, one can see that the CFL numbers for the 4th order SSP scheme are significantly higher than those of the Shu-RK3, but at the expense of more stages. Fig. 5(b) shows the CFL number normalized with the number of stages. From the figure, it is clear that the computational effort for both schemes is comparable, as the two curves lie close. However, note that the five-stage SSP scheme is 4th order accurate, i.e. one order higher than the Shu-RK3 scheme.

5. Convergence studies

5.1. Linear advection equation

Numerical simulations using the new Spectral Difference scheme have been performed to solve the linear advection Eq. (18) with \( f(u) = (c_x u, c_y u) \), where \( c_x \) and \( c_y \) are the advection velocities in the \( x \) and \( y \) directions, respectively. The equation is solved for a rectangular domain \([−1, 1] \times [−1, 1]\), with periodic boundary conditions. The solution field has been initialized at \( t = 0 \) as \( u(x,y) = \sin(2\pi(x + y)) \). A convergence study has been conducted using different mesh sizes for the SD-RT_m scheme, with \( m = 1, 2 \) and 3. The meshes have been generated from structured meshes, as outlined in Section 4, and flux nodes in the elements are distributed as shown in Fig. 3. The advection velocities considered were \( c_x = \cos(\pi/2) \) and \( c_y = \sin(\pi/2) \). The upwind numerical flux was used at the cell interfaces. Hence, the situation is identical to the setup that was used for linear stability analysis, and we have verified experimentally CFL numbers obtained in Section 4.

Fig. 6 shows the \( L_1 \) error at \( t = 0.1 \) for different meshes \( (N) \) and SD-RT_m schemes in logarithmic scale, where \( N = \frac{1}{h} \) with \( h \) as the characteristic length of the mesh. Since the error associated with time integration is negligible compared to the error associated with the spacial discretization when small time step is taken, the time integration was done using the Shu-RK3 scheme instead of the high order SSP scheme. Tables 3–5 show \( L_1 \) errors and orders of accuracy for different meshes and SD-RT_m schemes. The order of accuracy achieved here for the SD-RT_m scheme is approximately \( m + 1 \), the optimal attainable order for smooth solutions.

5.2. 2D Isentropic vortex

We consider the 2D Euler equations in the conservative form as

\[
\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} + \frac{\partial \mathbf{g}(\mathbf{u})}{\partial y} = 0,
\]

where \( \mathbf{u}, \mathbf{f} \) and \( \mathbf{g} \) are given by

Fig. 5. Influence of CFL number on different skew angles for Shu-RK3 and the 4th order SSP time stepping schemes for SD-RT_2 spacial discretization.
\[
\begin{align*}
\mathbf{u} &= \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \\
\mathbf{f} &= \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E + p) \end{bmatrix}, \\
\mathbf{g} &= \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E + p) \end{bmatrix}, \end{align*}
\]  

\hspace{1cm} (29)

where \( \rho \) is the density of the fluid, \( u \) is the \( x \) velocity, \( v \) is the \( y \) velocity, \( p \) is the pressure, and \( E \) is the total energy. To close the system of equations, we use the ideal gas law for which the relation between pressure and energy is given as

\[
p = (\gamma - 1) \left( E - \frac{1}{2} \rho (u^2 + v^2) \right),
\]  

\hspace{1cm} (30)

where \( \gamma = 1.4 \) is the specific heat capacity ratio for air.

---

**Fig. 6.** Convergence plot for the linear advection test case: The variation of \( l_1 \) error with different meshes \( (N) \) and SD-RT\(_m\) discretizations in logarithmic scale.

**Table 3**
Linear advection test case: order of accuracy and the \( l_1 \) error for different meshes for SD-RT\(_1\) discretization.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( l_1 ) Error</th>
<th>Order of accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>8.64E–2</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>1.05E–2</td>
<td>1.9184</td>
</tr>
<tr>
<td>50</td>
<td>3.78E–3</td>
<td>1.9999</td>
</tr>
<tr>
<td>80</td>
<td>1.48E–3</td>
<td>1.9950</td>
</tr>
<tr>
<td>100</td>
<td>9.47E–4</td>
<td>2.0009</td>
</tr>
</tbody>
</table>

**Table 4**
Linear advection test case: order of accuracy and the \( l_1 \) error for different meshes for SD-RT\(_2\) discretization.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( l_1 ) Error</th>
<th>Order of accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.00E–2</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>3.79E–4</td>
<td>2.9790</td>
</tr>
<tr>
<td>50</td>
<td>8.23E–5</td>
<td>2.9896</td>
</tr>
<tr>
<td>80</td>
<td>2.02E–5</td>
<td>2.9887</td>
</tr>
<tr>
<td>100</td>
<td>1.03E–5</td>
<td>3.0184</td>
</tr>
</tbody>
</table>

**Table 5**
Linear advection test case: order of accuracy and the \( l_1 \) error for different meshes for SD-RT\(_3\) discretization.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( l_1 ) Error</th>
<th>Order of accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5.88E–4</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>7.86E–6</td>
<td>3.9276</td>
</tr>
<tr>
<td>50</td>
<td>1.07E–6</td>
<td>3.9037</td>
</tr>
<tr>
<td>80</td>
<td>1.59E–7</td>
<td>4.0564</td>
</tr>
<tr>
<td>100</td>
<td>6.61E–8</td>
<td>3.9335</td>
</tr>
</tbody>
</table>
To simulate the isentropic vortex using the SD-RT scheme, a rectangular domain \([-1,1] \times [-1,1]\), was initialised with the mean flow values \((\rho, p, u, v) = (1, 1, 0, 0)\) and the vortex is added via perturbations to the mean flow values [25]. The perturbations in \(u, v, T, S\) are given as

\[
\begin{align*}
\delta u &= -y \frac{\epsilon}{2\pi} e^{0.5(1-r^2)}, \\
\delta v &= x \frac{\epsilon}{2\pi} e^{0.5(1-r^2)}, \\
\delta T &= -\frac{(\gamma - 1)\epsilon^2}{8\gamma\pi^2} e^{(1-r^2)}, \\
\delta S &= 0,
\end{align*}
\]

where \(r^2 = x^2 + y^2\) and \(\epsilon = 5\) is the vortex strength. The exact solution of the Euler equations with the above initial conditions is the convection of the isentropic vortex with the mean velocity. Since the mean velocity is zero in both \(x\) and \(y\) directions, the vortex stays at the same position as time advances. The simulations were conducted using SD-RT\(_1\), SD-RT\(_2\) and SD-RT\(_3\) discretizations. The Shu-RK3 scheme was used for the time integration. A convergence plot is shown in Fig. 7 in which the errors are calculated at \(t = 0.1\). A plot of the density contours generated at \(t = 0.1\) using SD-RT\(_2\) scheme is shown in Fig. 8. Tables 6–8 show \(l_\infty\) errors and orders of accuracy for different meshes and SD-RT\(_m\) schemes. Here also we obtained the optimal order of accuracy.

6. Numerical experiments: 2D Euler equations

The Euler Eqs. (28) and (29), together with Eq. (30) have been solved numerically on an unstructured grid for the subsonic steady-state flow over the airfoil NACA0012 using the SD-RT scheme. The surface of the airfoil is approximated by cubic splines along with a non-linear mapping to the reference element for mesh elements on the boundary. The non-linear mapping used in the present implementation is quadratic, i.e. super-parametric for the SD-RT\(_1\) scheme, isoparametric for the SD-RT\(_2\) scheme, and sub-parametric by one order in the case of SD-RT\(_3\).

The simulations were done using SD-RT\(_1\), SD-RT\(_2\) and SD-RT\(_3\) schemes. The flux nodes in the reference element are distributed as shown in Fig. 3. The computational mesh (1440 elements) used for the simulation is shown in Fig. 9. Slip boundary condition has been used on the surface of the airfoil and free stream values on the outer boundary. Jameson’s H-CUSP flux [26] was used as the numerical flux at the cell interfaces. So far no limiters or otherwise addition of artificial diffusion has been implemented, as we consider smooth flow only.

Time relaxation is done using a backward Euler discretization (i.e. damped Newton iteration). Let \((U^{n+1} - U^n) = \Delta U^n\), and write the implicit scheme

\[
\left( I - \Delta t \frac{dR(U^n)}{dU} \right) \Delta U^n = \Delta t R(U^n),
\]

Fig. 7. Convergence plot for the Isentropic vortex test case: The variation of \(l_\infty\) error with different mesh \((N)\) and SD-RT\(_m\), \(m = 1, 2 \text{ and } 3\) discretizations in logarithmic scale.
where the residual $R$ is defined in Eqs. (16) and (17), and $\frac{dR(U)}{dU}$ is the Jacobian matrix of the residual vector. The above linear system has to be solved at each step. We use implicit methods in particular for steady problems, where we require no time accuracy, but rather large time steps. The step size is made proportional in each cell to a local approximation to the spectral radius of the inviscid flux Jacobian, where the constant of proportionality is the CFL number of the implicit scheme. Note that as $\Delta t \rightarrow \infty$, we obtain Newton iterations.

The basic parameters to solve the linear system using an implicit scheme are the Jacobian matrix $\frac{dR(U)}{dU}$, the solution methodology and the preconditioning of the system. In our implementation we use an exact differentiation of the residual in order to allow asymptotic quadratic convergence, once a sufficiently good solution approximation has been reached. We use a heuristic time step control in the damped Newton/backward Euler approach, such that after a small number of start-up iterations...
The CFL number of the implicit scheme increases to infinity, as the residual is reduced. At the same time the relative accuracy for solution of the linear systems is increased. Furthermore, for small CFL numbers (typically CFL < 1000), the Jacobian matrix is frozen for a number of iterations (typically 3), while for larger CFL numbers it is recomputed at each iteration. The linear system is solved using the restarted GMRES(m) algorithm [27], where we typically restart after \( m = 30 \) Krylov vectors. The preconditioning of the system is done using Incomplete LU (ILU) factorization [28], typically ILU(2). Solution of the linear system is implemented using the PETSc library [29].
Fig. 12. Mach number contours generated using the SD-RT$_3$ scheme for $M_\infty = 0.3$ and $\alpha = 0^\circ$.

Fig. 13. Mach number contours generated using SD-RT$_1$ (top left), SD-RT$_2$ (top right) and SD-RT$_3$ (bottom) schemes for $M_\infty = 0.3$ and $\alpha = 0^\circ$. 
Table 9
 Drag coefficient ($c_d$) obtained using SD-RT$_m$ schemes for $M_1 = 0.3$ and $\alpha = 0^\circ$:

<table>
<thead>
<tr>
<th>SD-RT$_m$</th>
<th>$c_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SD-RT$_1$</td>
<td>9.3E-4</td>
</tr>
<tr>
<td>SD-RT$_2$</td>
<td>6.6E-5</td>
</tr>
<tr>
<td>SD-RT$_3$</td>
<td>4.7E-5</td>
</tr>
</tbody>
</table>

Fig. 14. Convergence of the residual for the subsonic lifting flow over NACA airfoil: residual vs. number of Jacobian evaluations for SD-RT$_m$ schemes.

Fig. 15. Convergence of the residual for the subsonic lifting flow over NACA airfoil: residual vs. cumulative number of GMRES iterations for SD-RT$_m$ schemes.

Fig. 16. Mach number contours generated using the SD-RT$_3$ scheme for $M_1 = 0.4$ and $\alpha = 5^\circ$. 

6.1. Test case 1: subsonic non-lifting flow over NACA0012 airfoil

We considered free stream Mach number $M_1 = 0.3$ and angle of attack $\alpha = 0^\circ$ for the non-lifting flow case. Figs. 10 and 11 show the decay of the residual against the number of Jacobian (Eq. (32)) evaluations and against the accumulated number of GMRES iterations, respectively for SD-RT1, SD-RT2 and SD-RT3 schemes. The run-time depends on various parameters such as time step control, number of start-up iterations, etc. Different settings were tried by varying parameters, and the convergence plots shown correspond to best-practice settings. Fig. 12 shows the contours of Mach number around the airfoil generated using the SD-RT3 scheme. Even with a coarse mesh, the Mach contours are well captured. Note that our postprocessing renders the solution in each triangle individually, and thus the contour lines are discontinuous across mesh elements. Smooth contours are thus indicative of adequate resolution. Fig. 13 shows the Mach number contours around the leading edge of the airfoil generated using SD-RT1, SD-RT2 and SD-RT3 schemes. Increasingly better resolution with higher polynomial degree is obvious from the figure. Table 9 shows the drag coefficient ($c_d$) obtained using the SD-RT scheme, which is approaching zero as the polynomial degree increases. Note that for $m = 3$, the convergence rate is perhaps somewhat reduced, probably because the non-linear mapping for the curved elements is only quadratic, and furthermore the singularity at the trailing edge of the airfoil makes this test case less than optimal for pure $p$-extension.

Table 10

<table>
<thead>
<tr>
<th>SD-RTm</th>
<th>$c_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SD-RT1</td>
<td>3.8E-3</td>
</tr>
<tr>
<td>SD-RT2</td>
<td>8.4E-4</td>
</tr>
<tr>
<td>SD-RT3</td>
<td>5.8E-4</td>
</tr>
</tbody>
</table>

Fig. 17. Mach number contours generated using SD-RT1 (top left), SD-RT2 (top right) and SD-RT3 (bottom) schemes for $M_1 = 0.4$ and $\alpha = 5^\circ$. 
6.2. Test case 2: subsonic lifting flow over NACA0012 airfoil

In the second test case, we consider free stream Mach number $M_{\infty} = 0.4$ and angle of attack $\alpha = 5^\circ$. Figs. 14 and 15 show the decay of the residual against the number of Jacobian (Eq. (32)) evaluations and against the accumulated number of GMRES iterations, respectively for SD-RT1, SD-RT2 and SD-RT3 schemes. Fig. 16 shows the contours of Mach number around the airfoil generated using the SD-RT3 scheme. The Mach number contours around the leading edge of the airfoil, generated using SD-RT1, SD-RT2 and SD-RT3 schemes are shown in Fig. 17. Again, increasing resolution with higher polynomial degree is clearly visible. Table 10 shows the drag coefficient ($cd$) obtained using the SD-RT scheme. Convergence may be observed as the values tend to zero. Again for $m = 3$, the convergence rate is perhaps somewhat reduced, due to the reasons mentioned for the previous test case in Section 6.1.

7. Conclusions and future work

A new variant of the Spectral Difference scheme which uses Raviart–Thomas elements (SD-RT) has been implemented and validated. Linear stability analysis has been extended to identify stable flux node configurations for the SD-RT$_m$, $m = 1, 2$ and 3, schemes for both regular and stretched meshes with high skew angles. Convergence studies have been conducted with the SD-RT scheme for the linear advection and the isotropic vortex problems for the 2D case. The results show that full order of convergence is achieved. This motivated us to perform simulations using the new scheme for the more complex case of 2D flow over an airfoil. The Euler equations have been solved around the NACA0012 airfoil for subsonic flow cases. Promising results were obtained using the SD-RT scheme.

Future work should aim at finding a general set of stable flux nodes for SD-RT$_m$, $m \geq 4$ schemes. For solving real life compressible fluid flow problems, the scheme should be equipped with methodologies to deal with discontinuities, such as shock waves, in the flow. Furthermore, the scheme needs to be extended to solve the compressible Navier–Stokes equations. These challenges are left for future investigations.

Acknowledgments

Financial support from the Deutsche Forschungsgemeinschaft (German Research Association) through Grant GSC 111, and by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under Grant number FA8655-08-1-3060, is gratefully acknowledged. The US Government is authorized to reproduce and distribute reprints for Governmental purpose notwithstanding any copyright notation thereon.

References