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Ohm's law for plasma in general relativity and Cowling's theorem

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Abstract The general-relativistic Ohm's law for a two-component plasma which includes the gravitomagnetic force terms even in the case of quasi-neutrality has been derived. The equations that describe the electromagnetic processes in a plasma surrounding a neutron star are obtained by using the general relativistic form of Maxwell equations in a geometry of slow rotating gravitational object. In addition to the general-relativistic effect first discussed by Khanna and Camenzind (Astron. Astrophys. 307:665, 1996) we predict a mechanism of the generation of azimuthal current under the general relativistic effect of dragging of inertial frames on radial current in a plasma around neutron star. The azimuthal current being proportional to the angular velocity ω of the dragging of inertial frames can give valuable contribution on the evolution of the stellar magnetic field if ω exceeds $2.7 \times 10^{17}(n/\sigma)$ s^{-1} (n is the number density of the charged particles, σ is the conductivity of plasma). Thus in general relativity a rotating neutron star, embedded in plasma, can in principle generate axial-symmetric magnetic fields even in axisymmetry. However, classical Cowling's antidynamo theorem, according to which a stationary axial-symmetric magnetic field can not be sustained against ohmic diffusion, has to be hold in the general-relativistic case for the typical plasma being responsible for the rotating neutron star.

Keywords MHD · Plasmas · General relativity · Magnetic field

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1 Introduction

The discovery of pulsars, characterized by the emission of radio pulses at regular intervals has lead to the belief that pulsars are compact rotating neutron stars with extremely large frozen-in magnetic fields. The high conductivity of the stellar interior ensures conservation of magnetic flux during collapse. For this reason, it is believed the strength of initial magnetic field will increase quadratically with the decrease of the linear dimensions and reach, as a rule, up to 10^{12} G in the final state for the young neutron star. The time evolution of such strong magnetic field has been the subject of much discussion over the years and the reviews (e.g. Reisenegger 2009; Geppert 2009; Bhattacharya 2002; Lamb 1991; Chanmugam 1992; Phinney and Kulkarni 1994) provide the present understanding on the decay of magnetic fields in isolated neutron stars.

Recent observation of pulsars and their statistical analysis seem to imply that the evolution of magnetic fields of isolated pulsars is still relatively open and the evidence supporting that they do not undergo or undergo significant magnetic field decay is not conclusive due to large error bars. However it is well-known, in general, a dynamo, when the motions of a conducting fluid are able to sustain or increase a stellar magnetic field, needs. But for a number of simple geometries no dynamo is possible and according to the famous Cowling's antidynamo theorem (Cowling 1934) the stationary axial-symmetric magnetic field can not be self maintained.

The situation may be different in a general-relativistic context. That is why the kinematic evolution of axisymmetric magnetic and electric fields has been recently investigated in Kerr geometry (Khanna and Camenzind 1994, 1996; Brandenburg 1996; Nunez 1996, 1997). Interesting other types of fast dynamo mechanisms, based on stretching

flux tubes in Riemannian conformal manifolds, have been also recently obtained by Garcia de Andrade (2007, 2008). However it was found in (Khanna and Camenzind 1994, 1996; Brandenburg 1996; Montelongo-Garcia and Zannias 2006) that a magnetic field can not be also sustained against ohmic diffusion in general relativistic case as in flat space-time one. No support was found for the possibility that the general relativistic effects could lead to self-excited axisymmetric solutions. Nunez (1997) states that the gravitomagnetic potential in the Kerr metric couples the equations of the magnetic flux and current, rendering invalid the argument used in the proof of Cowling's antidynamo theorem. In this respect the magnetic field evolution, especially general relativistic contribution to it, around and in a supermassive stars is extremely significant component in the recent models of pulsars and rotating neutron stars.

Electrodynamics in a four-dimensional spacetime "feels" inertial and gravitational effects via the metric dependent constitutive (spacetime and material) relations. For conductors, this is manifest in the covariant generalization of Ohm's law see e.g. Ahmedov (1999b). Relativistic version of the generalized Ohm's law for plasma can be found e.g. in Ardavan (1976), Blackman and Field (1993), Gedalin (1996), Khanna (1998), Kremer and Patsko (2003), Meier (2004), Kandus and Tsagas (2008), Koide (2009).

In this paper the classical derivation of Cowling's theorem is repeated using general relativistic electromagnetic equations governing two-component plasma in the background geometry of stationary gravitational body. This set of equations has general relativistic Ohm's law for plasma including new general-relativistic gravitomagnetic terms, which may be fundamental for the study the generation and evolution of stellar magnetic field. We obtain a nonvanishing general relativistic expression for the circulation of electric current, which is due to the general relativistic frame dragging effect on the radial electric current flowing in plasma in the vicinity of a rotating neutron star. Thus we show, that in axisymmetry, the gravitomagnetic effects can drive currents and generate magnetic fields, even without taking into account the possibility of turbulence and α-dynamo effect. But from our evaluations for the typical astrophysical plasma the generated magnetic field may be extremely weak in order to be taken into account.

2 Ohm's law for plasma in general relativity

One may consider the electrons and ions in plasma as separate fluids which are interacting with each other through collisions. This two-fluid model is also essential for deriving the general relativistic Ohm's law for plasma and for describing the different effects being responsible for the generation of the electromagnetic fields. To find this law we consider the equations describing the motion of the individual

components of the plasma, i.e. the linearized equations of motion of electrons and ions

$$c^2 u^\alpha_{(e);\sigma} u^\sigma_{(e)} = -\frac{e}{m} F^{\alpha\beta} u_{(e)\beta} - \nu_1 (v^\alpha_{(e)} - v^\alpha_{(i)})$$
$$- \frac{\Lambda^{-1/2}}{mN_e} \overset{\perp}{\nabla}_\alpha \tilde{p}_e, \quad (1)$$

$$c^2 u^\alpha_{(i);\sigma} u^\sigma_{(i)} = \frac{e}{M} F^{\alpha\beta} u_{(i)\beta} - \nu_2 (v^\alpha_{(i)} - v^\alpha_{(e)})$$
$$- \frac{\Lambda^{-1/2}}{M_i N_i} \overset{\perp}{\nabla}_\alpha \tilde{p}_i, \quad (2)$$

where ν_1 and ν_2 are the collision frequencies, m and M_i are the mass of electron and ion, the subscripts i and e denote ion and electron quantities, respectively, semicolon is the covariant derivative, c is the speed of light, $(\Lambda^{1/2} p_e) = \tilde{p}_e$, the term $\Lambda^{-1/2} \overset{\perp}{\nabla}_\alpha (\Lambda^{1/2} p_{e,i})$ is due to the change of the pressure $p_{e,i}$ of electron and ion components and $\overset{\perp}{\nabla}_\alpha$ denotes the spatial part of covariant derivative, $F_{\alpha\beta}$ is the tensor of electromagnetic field.

The gravitational field is assumed to be stationary that is space-time metric $g_{\alpha\beta}$ admits a timelike Killing vector $\xi^\alpha_{(t)}$ that is $\pounds_{\xi_t} g_{\alpha\beta} = 0$ ($\pounds_{\xi_t}$ denotes the Lie derivative with respect to $\xi^\alpha_{(t)}$, $\Lambda = -\xi^\alpha_{(t)} \xi_{(t)\alpha}$). The gravitational field, represented by the metric tensor, is assumed to be generated by outside gravitational source. The plasma itself is expected to generate a much weaker gravitational field.

After doing some algebraic transformations equations (1) and (2) can be written in the form:

$$c\partial_T v^\alpha_{(e)} = -\frac{e}{m} F^{\alpha\beta} u_\beta - \frac{e}{mc} (F^{\alpha\sigma} + F^{\rho\sigma} u_\rho u^\alpha) v_{(e)\sigma}$$
$$- \nu_1 (v^\alpha_{(e)} - v^\alpha_{(i)}) - c^2 w^\alpha - 2c v^\beta_{(e)} A^\alpha_{.\beta}$$
$$- \frac{\Lambda^{-1/2}}{mN_e} \overset{\perp}{\nabla}_\alpha \tilde{p}_e, \quad (3)$$

$$c\partial_T v^\alpha_{(i)} = \frac{e}{M_i} F^{\alpha\beta} u_\beta + \frac{e}{M_i c} (F^{\alpha\sigma} + F^{\rho\sigma} u_\rho u^\alpha) v_{(i)\sigma}$$
$$- \nu_2 (v^\alpha_{(i)} - v^\alpha_{(e)}) - c^2 w^\alpha - 2c v^\beta_{(i)} A^\alpha_{.\beta}$$
$$- \frac{\Lambda^{-1/2}}{M_i N_i} \overset{\perp}{\nabla}_\alpha \tilde{p}_i. \quad (4)$$

Here u_β is the four-velocity of the proper frame, $u_{\mu;\nu} = A_{\mu\nu} - D_{\mu\nu} + w_\mu u_\nu$, $A_{\beta\alpha} = u_{[\alpha,\beta]} + u_{[\beta} w_{\alpha]}$ is the relativistic rate of rotation, $w_\alpha = u_{\alpha;\beta} u^\beta$ is the absolute acceleration, $[\cdots]$ denotes the antisymmetrization, ∂_T denotes the time derivative (Vladimirov 1982; Antonov et al. 1978), $D_{\mu\nu} = \partial_T h_{\mu\nu}/2$[1] is the tensor of deformation velocities, $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$; the relative velocities of electrons and

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[1]It is assumed that the spacetime is stationary where $D_{\mu\nu} = 0$.

ions are

$$v_{(e)}^\alpha/c = \sqrt{1 - v_{(e)}^2/c^2}\, u^\alpha - u_{(e)}^\alpha, \tag{5}$$

$$v_{(i)}^\alpha/c = \sqrt{1 - v_{(i)}^2/c^2}\, u^\alpha - u_{(i)}^\alpha. \tag{6}$$

The mean velocity v^α, the current density J^α and the mass density ρ_m are

$$v^\alpha = \frac{(M_i N_i v_{(i)}^\alpha + m N_e v_{(e)}^\alpha)}{M_i N_i + m N_e}, \tag{7}$$

$$J^\alpha = e(N_i v_{(i)}^\alpha - N_e v_{(e)}^\alpha), \tag{8}$$

$$\rho_m = M_i N_i + m N_e. \tag{9}$$

For simplicity we here assume that $N_e \approx N_i = N$ (i.e. the charge neutrality) and since $M_i \gg m$, we find that

$$v^\alpha = v_{(i)}^\alpha + \frac{m}{M_i} v_{(e)}^\alpha,$$

$$J^\alpha = Ne\left\{ v^\alpha - \left(\frac{m}{M_i} + 1\right) v_{(e)}^\alpha \right\}. \tag{10}$$

Hence

$$v_{(e)}^\alpha = -\frac{J^\alpha}{Ne} + v^\alpha, \tag{11}$$

$$v_{(i)}^\alpha = v^\alpha + \frac{m}{\rho_m e} J^\alpha. \tag{12}$$

From putting the derived expressions for $v_{(i)}^\alpha$ and $v_{(e)}^\alpha$ (11) and (12) into formulae (3) and (4), we obtain

$$\partial_T \left(v^\alpha - \frac{J^\alpha}{Ne} \right)$$
$$= -\frac{e}{mc} F^{\alpha\beta} u_\beta - \frac{\nu_1}{c}(v_{(e)}^\alpha - v_{(i)}^\alpha)$$
$$- \frac{e}{mc^2}(F^{\alpha\sigma} + F^{\rho\sigma} u_\rho u^\alpha)\left(v_\sigma - \frac{J_\sigma}{Ne} \right) - cw^\alpha$$
$$- 2\left(v^\beta - \frac{J^\beta}{Ne} \right) A_{.\beta}^\alpha - \frac{\Lambda^{-1/2}}{cmN_e} \overset{\perp}{\nabla}_\alpha \tilde{p}_e, \tag{13}$$

$$\partial_T \left(v^\alpha + \frac{mJ^\alpha}{\rho_m e} \right)$$
$$= \frac{e}{M_i c} F^{\alpha\beta} u_\beta - \frac{\nu_2}{c}(v_{(i)}^\alpha - v_{(e)}^\alpha)$$
$$+ \frac{e}{M_i c^2}(F^{\alpha\sigma} + F^{\rho\sigma} u_\rho u^\alpha)\left(v_\sigma + \frac{mJ_\sigma}{\rho_m e} \right) - cw^\alpha$$
$$- 2\left(v^\beta + \frac{mJ^\beta}{\rho_m e} \right) A_{.\beta}^\alpha - \frac{\Lambda^{-1/2}}{cM_i N_i} \overset{\perp}{\nabla}_\alpha \tilde{p}_i. \tag{14}$$

In order to obtain the generalized Ohm's law for the plasma as a single fluid we combine the equations of mo-

tion and substract (13) from (14), and get approximately

$$F^{\alpha\beta} u_\beta + \frac{1}{c}(F^{\alpha\sigma} + F^{\rho\sigma} u_\rho u^\alpha) v_\sigma$$
$$= -\frac{\Lambda^{-1/2}}{Ne} \overset{\perp}{\nabla}_\alpha \tilde{p}_e$$
$$+ \frac{J^\alpha}{\sigma} - \frac{1}{Nec}(F^{\alpha\sigma} + F^{\rho\sigma} u_\rho u^\alpha) J_\sigma$$
$$- \frac{mc}{Ne^2} \partial_T J^\alpha + \frac{2mc}{Ne^2} J^\beta A_{.\beta}^\alpha. \tag{15}$$

Here $\nu = \nu_1 + \nu_2$, $\sigma = \frac{Ne^2}{m\nu}$ is the electrical conductivity in the presence of a constant electric field and zero magnetic field.

We follow the description of the generalized Ohm's law (15) from the review paper Brandenburg and Subramanian (2005). The first term on the right hand side of (15), being produced by the electron pressure gradient, is the Biermann battery term (Biermann 1950). For example it may provide the source term for the thermally generated electromagnetic fields (Mestel and Roxburgh 1962). The next two terms on the right hand side of (15) are the usual Ohmic term and the Hall electric field, which arises due to a nonvanishing Lorentz force. The next term on the right hand side is the inertial term, which can be neglected if the macroscopic time scales are large compared to the plasma oscillation periods. And finally the last term on the right hand side of (15) appears due to the Coriolis force and dragging of inertial frames effects on the conduction current.

In neutron stars, the presence of strong magnetic fields, could make the Hall term important, especially in their outer regions, where there are also strong density gradients. The Hall effect in neutron stars can lead to magnetic fields undergoing a turbulent cascade (Goldreich and Reisenegger 1992). By analogy with the vorticity equation in ordinary hydrodynamics, Goldreich and Reisenegger (1992) conjectured that the transfer of magnetic energy from large to small scales proceeds in a similar way to ordinary turbulence. However, the analogy of the Hall induction equation with the vorticity equation is not complete, and the conjecture remained to be confirmed by multidimensional numerical simulations. It can also lead to a nonlinear steepening of field gradients (Vainshtein et al. 2000) for purely toroidal fields, and hence to enhanced magnetic field dissipation. Vainshtein et al. (2000) proposed a mechanism for the fast dissipation of magnetic field based on the Hall drift in stratified media. They correctly pointed out that Hall currents are able to create current sheets (which are sites for efficient dissipation) and that the evolution of the toroidal field resembles the Burgers equation. The same Burgers-like equation is applicable even to non-stratified media, but in a spherical shell (Pons and Geppert 2007, 2010), in which the Hall term

in the induction equation tends to create current sheets instead of ordinary turbulence. Rheinhardt and Geppert (2002) showed by a linear analysis that, in a one-component (electron) plasma, a large-scale background magnetic field may become unstable to smaller scale perturbations. This Hall-drift induced instability occurs when the magnetization parameter is high and the background field has enough curvature. Since these conditions may be realized in the crust of a neutron star, the problem of the Hall effect became interesting not only from the magnetohydrodynamic point of view but also for the astrophysics community. Although the Hall-drift is a non-dissipative process, the growth of small-scale magnetic field components modifies the overall magnetic field structure and opens the possibility of more rapid field decay than pure ohmic dissipation would predict.

For a stationary plasma $v_\sigma = 0$ and in the steady state $\partial_T j^\alpha = 0$, this equation becomes

$$j^\alpha = \frac{Ne^2}{m\nu}\left\{F^{\alpha\beta}u_\beta + \frac{\Lambda^{-1/2}}{Ne}\overset{\perp}{\nabla}_\alpha\tilde{p}_e\right\}$$
$$- \frac{e}{m\nu c}(F^{\alpha\sigma} + F^{\rho\sigma}u_\rho u^\alpha)J_\sigma + \frac{2c}{\nu}J^\beta A^\alpha_{.\beta}. \quad (16)$$

It can be written as

$$\frac{J_\alpha}{\sigma} = F_{\alpha\beta}u^\beta - R_H(F_{\alpha\sigma} + F_{\rho\sigma}u^\rho u_\alpha)J^\sigma$$
$$+ R_{gg}J^\beta A_{\alpha\beta} + \frac{\Lambda^{-1/2}}{Ne}\overset{\perp}{\nabla}_\alpha\tilde{p}_e. \quad (17)$$

This is the generalized Ohm's law for two-component plasma in general relativity. Here

$$R_H = \frac{1}{Nec}, \quad R_{gg} = \frac{2mc}{Ne^2}, \quad (18)$$

obviously R_H is the Hall constant, R_{gg} is the parameter for the plasma called as galvano-gravitomagnetic one.

Khanna (1998) has derived the general relativistic Ohm's law for two-component plasma and concluded that it has no new terms as compared with special relativity in the limit of quasi-neutral plasma. From our point of view the gravito-magnetic terms did not appear in Ohm's law most probably as a consequence of the magnetohydrodynamic approximation used in Khanna (1998). The first two terms in the right hand side of (17) are standard classical terms which include the general relativistic contributions. The third term in the right hand side of (17) has been discussed for the conduction current in conductors (Ahmedov 1998, 1999a, 1999b). It has purely relativistic nature and is caused by the effect of gravitomagnetic force on the electric current flowing in the plasma. Also it has recently been obtained for plasma by Kandus and Tsagas (2008), see to the third term under the square bracket in the right hand side of their equation (56).

3 Space charge density in two-component plasma

As a consequence of (17) for a plasma without conduction current $J^\alpha = 0$, electric field $E^\alpha = F^{\alpha\beta}u_\beta = -\overset{\perp}{\nabla}_\alpha\tilde{p}_e$. This electric field inside a plasma in a gravitational field is a well-known phenomenon in the magnetosphere and in stellar interiors. In the plasma magnetosphere, gravity acts strongly on ion constituent while the electron constituent tends to escape to infinity. An upward-directed electric field is set up that prevents the electrons from escaping, thus charge neutrality is maintained within magnetospheric medium. In the magnetosphere, the strength of this upward-directed electric field depends on the mass of ion constituent and on the electron temperature.

The number density of each particle species varies exponentially as $-\tilde{\mu}/kT$ (Ehlers 1971), where $\tilde{\mu} = \Lambda^{1/2}\mu$ is the gravito-electro-chemical potential including the rest mass energy, k is Boltzmann's constant, and T is the temperature.

To preserve electrical quasi-neutrality the variation of the densities of ions and electrons with height must be essentially the same for each constituent, giving therefore

$$\tilde{\mu}_i/kT_i = \tilde{\mu}_e/kT_e \quad (19)$$

and consequently

$$M_i c^2 w_\alpha + eE_\alpha/T_i = mc^2 w_\alpha - eE_\alpha/T_e, \quad (20)$$

where E_α is the electric field required to maintain charge neutrality in the presence of gravitational field. The field E_α is then, for single charged ions,

$$E_\alpha = \frac{w_\alpha(mc^2 T_i - M_i c^2 T_e)}{e(T_e + T_i)}, \quad (21)$$

or for physically realisable case where T_e is comparable with T_i and $M_i \gg m$

$$E_\alpha = \frac{M_i c^2 w_\alpha}{e}\frac{T_e}{T_i + T_e}. \quad (22)$$

We see that the induced electric field in a plasma is, to an excellent approximation, principally a function of the ion mass and electron temperature. In the case of an isothermal plasma composed entirely of electrons and positrons, electric field $E_\alpha = 0$.

If we suppose that the material relations between inductions and fields have linear character i.e.

$$H_{\alpha\beta} = \frac{1}{\mu}F_{\alpha\beta} + \frac{1-\epsilon\mu}{\mu}(u_\alpha F_{\sigma\beta} - u_\beta F_{\sigma\alpha})u^\sigma, \quad (23)$$

$$F_{\alpha\beta} = \mu H_{\alpha\beta} + \frac{\epsilon\mu-1}{\epsilon}(u_\alpha H_{\sigma\beta} - u_\beta H_{\sigma\alpha})u^\sigma \quad (24)$$

and use the generalized Ohm's law for plasma then one can easily derive the general formula for the space charge distribution inside plasma

$$
\rho_0 = \frac{\epsilon\mu R_H}{c}J^2 + \frac{1}{4\pi}\left\{\left(\frac{\epsilon}{\sigma}J^\alpha\right)_{;\alpha} - \epsilon w^\alpha\frac{\Lambda^{-1/2}}{Ne}\overset{\perp}{\nabla}_\alpha\tilde{p}_e\right.
$$
$$
+ \left[\epsilon^2\mu R_H\left(\frac{1}{\sigma}J^2 + \frac{\Lambda^{-1/2}}{Ne}J^\nu\overset{\perp}{\nabla}_\nu\tilde{p}_e\right)u^\alpha\right]_{;\alpha}
$$
$$
- \epsilon R_{gg}A_{\alpha\beta}w^\alpha J^\beta + g^{\alpha\beta}(\epsilon R_{gg}J^\nu A_{\alpha\nu})_{;\beta} - \frac{\epsilon}{\sigma}w^\alpha J_\alpha
$$
$$
+ g^{\alpha\beta}\left(\epsilon\frac{\Lambda^{-1/2}}{Ne}\overset{\perp}{\nabla}_\alpha\tilde{p}_e\right)_{;\beta}
$$
$$
+ H^{\alpha\beta}[A_{\beta\alpha} + \epsilon\mu R_H w_\alpha J_\beta + (\epsilon\mu R_H J_\alpha)_{,\beta}]\right\} \quad (25)
$$

from the general relativistic Maxwell equations

$$
e^{\alpha\beta\mu\nu}F_{\beta\mu,\nu} = 0, \quad H^{\alpha\beta}{}_{;\beta} = \frac{4\pi}{c}J^\alpha, \quad J^\alpha = c\rho_0 u^\alpha + j^\alpha. \tag{26}
$$

Here $H_{\alpha\beta}$ is the tensor of electromagnetic induction, ϵ and μ are the parameters for the plasma.

Even in the case when there are no any currents flowing in plasma ($J^\alpha = 0$) the nonvanishing space charge

$$
\rho_0 = \frac{1}{4\pi}\left\{g^{\alpha\beta}\left(\epsilon\frac{\Lambda^{-1/2}}{Ne}\overset{\perp}{\nabla}_\alpha\tilde{p}_e\right)_{;\beta}\right.
$$
$$
\left. - \epsilon w^\alpha\frac{\Lambda^{-1/2}}{Ne}\overset{\perp}{\nabla}_\alpha\tilde{p}_e + H^{\alpha\beta}A_{\beta\alpha}\right\} \tag{27}
$$

appears and is a sum of two contributions: one is due to the inner electric field (22) discussed above and second one is the general-relativistic generalization of the Goldreich-Julian charge density (Goldreich and Julian 1969) in plasma magnetosphere of rotating magnetized neutron star.

4 On Cowling's theorem in stationary gravitational field

According to the Cowling's theorem (Cowling 1934), steady plasma motions can not maintain a magnetic field that is confined to a finite region of space and possesses axial symmetry. Our task is to reformulate this theorem for the case when external gravitational field exists, i.e. to generalize to the general relativistic context the simple version of this theorem presented for example in Choudhuri (1998).

The metric of an asymptotically flat, stationary, axially symmetric spacetime around a rotating gravitating body (see, e.g. Landau and Lifshitz 1975) is considered. In spherical polar coordinates $x^0 = ct$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \varphi$,

we have

$$
ds^2 = -e^{2\Phi(r)}(cdt)^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2
$$
$$
+ r^2\sin^2\theta(d\varphi - \omega dt)^2, \tag{28}
$$

where $e^{\Phi(r)} = e^{-\Lambda(r)} = (1 - 2GM/c^2 r)^{1/2}$ is the gravitational redshift function of body (neutron star) of mass M, J is the angular momentum of a neutron star and G is the gravitational constant. The metric in (28) is the approximation of Kerr metric when the angular momentum is small. The presence of the nondiagonal component in metric in (28) results in the well known effect of dragging of inertial frames of reference (the Lense-Thirring effect) with the angular velocity

$$
\omega = \frac{2GJ}{c^2 r^3}. \tag{29}
$$

Here $\tilde{\omega} = \omega - \Omega$ is the angular velocity of the fluid as measured from the local free falling frame, Ω is the angular velocity of rotation of star relative to the distant observer.

The stationary and locally nonrotating "Zero Angular Momentum Observers" (ZAMO) (Bardeen et al. 1972; Thorne and Macdonald 1982; Thorne et al. 1986) are described by their four-velocity

$$
(u^\nu)_{\text{ZAMO}}\left\{\frac{1}{\sqrt{1-2M/r}}, 0, 0, \frac{\omega}{c\sqrt{1-2M/r}}\right\},
$$
$$
(u_\nu)_{\text{ZAMO}}\{-\sqrt{1-2M/r}, 0, 0, 0\}. \tag{30}
$$

The metric given by (28) has two Killing vectors which are responsible for stationarity and axial symmetry and can be chosen as

$$
\xi^\alpha_{(t)} = (1, 0, 0, 0), \qquad \xi^\alpha_{(\varphi)} = (0, 0, 0, 1),
$$
$$
\omega = -\xi_{(t)}\cdot\xi_{(\varphi)}/\xi_{(\varphi)}\cdot\xi_{(\varphi)}. \tag{31}
$$

The four-velocity of ZAMOs is connected with the Killing vectors according to formula

$$
(u^\alpha)_{\text{ZAMO}} = \Lambda^{-1/2}(\xi^\alpha_{(t)} + \omega\xi^\alpha_{(\varphi)}). \tag{32}
$$

Write a steady, axisymmetric magnetic field of star as the sum of a toroidal (i.e., azimuthal) component $\mathbf{B}^{\hat{\varphi}}$ and poloidal component $\mathbf{B}_p$ (which itself represents the sum of the radial and axial components in cylindrical polars)

$$
\mathbf{B} = B_{\hat{\varphi}}\mathbf{i}^{\hat{\varphi}} + \mathbf{B}_p, \tag{33}
$$

hats label the orthonormal components.

Because of the axisymmetry, the magnetic configuration in all meridional planes (through the axis of symmetry) is the same and must consist of closed field lines (see Fig. 1). In each meridional plane there must therefore exist at least one 0-type neutral point (N), where a poloidal component of magnetic field vanishes so that the field is purely azimuthal.
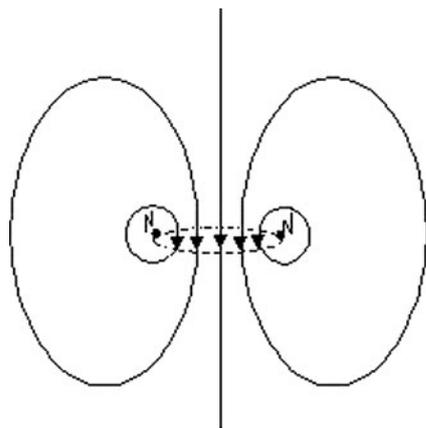
Fig. 1 Magnetic force lines of the axial-symmetric field in a meridional plane. *Dashed line* is the integration path through the neutral points N

Now, the general relativistic Ohm's law in the form (17) can be integrated around the closed line of force (C) through the neutral points (N) to give

$$\oint \frac{\Lambda^{1/2}}{\sigma} J_\alpha dx^\alpha$$

$$= \oint \Lambda^{1/2} F_{\alpha\beta} u^\beta dx^\alpha + \oint \Lambda^{1/2} R_{gg} J^\beta A_{\alpha\beta} dx^\alpha$$

$$- \oint \Lambda^{1/2} R_H (F_{\alpha\sigma} + F_{\rho\sigma} u^\rho u_\alpha) J^\sigma dx^\alpha. \tag{34}$$

The component due to the gradient of pressure of charged particles is curl-free and the circulation of the last term in the right hand side of (17) vanishes[2] if we neglect the inhomogeneity of N. The terms on the right hand side of (17) describe, in order, the effects of Ohmic decay, Hall effect and the effect of dragging of inertial frames. If the second and third terms of the right-hand side of (17) can be neglected, this equation is reduced to the standard form (having general-relativistic corrections due to $\Lambda^{1/2}$) already used by many authors.

Next we evaluate the conditions under which the Hall current becomes important. Comparing the rough magnitudes of the second term on the right-hand side and the left hand side of (34), we cannot neglect the second term if magnetic field is

$$B \geq \frac{nec}{\sigma} \approx 1.44 \frac{n}{\sigma} G. \tag{35}$$

However, the Hall drift does not work under a specific field configuration, for example, if the toroidal field never appears, the evolution of the pure poloidal field can not be influenced by the Hall term.

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[2]In general, general relativistic Ohm's law contains contributions due to the thermoelectric effects which are also curl-free.

One can evaluate the conditions when the gravitomagnetic term becomes valuable. The rough comparison of the left-hand side and the last term on the right-hand side of (34) gives that the gravitomagnetic term cannot be neglected if

$$\omega \geq \frac{ne}{2m\sigma} \approx 2.7 \times 10^{17} \frac{n}{\sigma} \text{ s}^{-1}. \tag{36}$$

It shows that in principle the Hall drift is much stronger than the gravitomagnetic one and in this respect more important for the astrophysical processes. Only in the exceptional case when the Hall drift does not work one can discuss the gravitomagnetic effect.

The four-velocity of plasma can be decomposed in the form

$$u^\alpha = \frac{u^\alpha_{(0)} + v^\alpha/c}{\sqrt{1 - v^2/c^2}}$$

$$\approx u^\alpha_{(0)} + v^\alpha/c$$

$$= (u^\alpha)_{\text{ZAMO}} - \Lambda^{-1/2} \omega \xi^\alpha_{(\varphi)} + v^\alpha/c, \tag{37}$$

where four-velocity field $u^\alpha_{(0)} = \Lambda^{-1/2} \xi^\alpha_{(t)}$ is parallel to the timelike Killing vector.

Let us calculate the value of the integral

$$\oint \Lambda^{1/2} F_{\alpha\beta} v^\beta dx^\alpha = \oint \Lambda^{1/2} (E_\beta v^\beta)(u_\alpha)_{\text{ZAMO}} dx^\alpha$$

$$- \oint \Lambda^{1/2} ((u_\beta)_{\text{ZAMO}} v^\beta) E_\alpha dx^\alpha$$

$$+ \oint \Lambda^{1/2} e_{\alpha\beta\mu\nu} (u^\mu)_{\text{ZAMO}} B^\nu v^\beta dx^\alpha \tag{38}$$

when $\alpha = 3$.

In this case the first integral is equal to zero according to (30). From formulae (32) and (37) one can get

$$\Lambda^{1/2} ((u^\beta)_{\text{ZAMO}} v_\beta) = \omega \xi^\beta_{(\varphi)} v_\beta . \tag{39}$$

It is meant that in general due to 'the dragging of the reference frame' the second term in (38) does not vanish since the electric field has two contributions: one is proportional to the electric current and second one being produced by the stellar magnetic field is proportional to angular velocity. Thus the last one is also negligible in the linear in angular velocity of rotation approximation.

The third integral

$$\oint \Lambda^{1/2} r \sin\theta (B^{\hat\theta} v^{\hat r} - B^{\hat r} v^{\hat\theta}) d\varphi = 0 \tag{40}$$

in the right hand side of (38) which is the induction term due to the hydrodynamic motion of plasma with velocity v^β is identically vanishing since $B^{\hat r} = B^{\hat\theta} = 0$ along a contour through the neutral points N.

By using of Maxwell equations (26) and Stoke's theorem (Misner et al. 1973) one can show that

$$\oint \Lambda^{1/2} F_{\alpha\beta} u^{\beta}_{(0)} dx^{\alpha} = -\frac{1}{2} \int (\pounds_{\xi_t} F_{\alpha\beta}) dS^{\alpha\beta} = 0. \quad (41)$$

Since the electromagnetic field is stationary by assumption, i.e. $\pounds_{\xi_t} F_{\alpha\beta} = 0$, then the first term under the integral on the right-hand side of (34) vanishes.

Thus the integral of Ohm's law (34) reduces to

$$\oint \frac{\Lambda^{1/2}}{\sigma} J_{\alpha} dx^{\alpha}$$

$$= -\oint \Lambda^{1/2} R_H (F_{\alpha\sigma} + F_{\rho\sigma} u^{\rho} u_{\alpha}) J^{\sigma} dx^{\alpha}$$

$$+ \oint \Lambda^{1/2} R_{gg} \hat{\jmath}^{\beta} A_{\alpha\beta} dx^{\alpha}$$

$$+ \oint \Lambda^{1/2} \omega (\xi_{\varphi}^{\beta} v_{\beta}) E_{\alpha} dx^{\alpha}. \quad (42)$$

Hence exact maintenance of the field in the principle is possible. The last term on the right hand side of (42) is the one first discussed by Khanna and Camenzind (1996). It disappears in the linear approximation in ω if there are no any conduction currents in the plasma.

The physical interpretation of the other terms on the right hand side of (42) is as follows. On one hand changes in the field are due to the motion, which transports the field lines from point to point, and to the finite conductivity, which permits the field to diffuse from point to point and so decay. But the Hall and gravitomagnetic force effects on radial current produce the current in azimuthal direction which creates new field lines. Thus there is, in principle, the mechanism to balance the diffusive decay of the field and a steady state is possible. However the importance of the Hall and gravitomagnetic effects depends on the model since these effects does not work under a specific field configurations. For more details about the effect of Hall drift in the neutron stars, see e.g. Pons and Geppert (2007, 2010), Jones (1988), Rheinhardt and Geppert (2002), Goldreich and Reisenegger (1992), Naito and Kojima (1994), Muslimov (1994).

This result means that the stationary axisymmetric electromagnetic field in general relativity, in fact, can be supported by the relativistic rate of rotation $A_{\alpha\beta}$. For example, for the plasma embedded in the Schwarzschild space time with $A_{\alpha\beta} \equiv 0$ the classical Cowling's antidynamo theorem is valid. But for the space-time of slow rotating compact object with nonvanishing nondiagonal components of metric tensor, the relativistic rate of rotation $A_{\alpha\beta}$ is nonzero. The radial current experiences effect of the gravitomagnetic force and therefore according to (17) we have the following value for the gravitomagnetically generated azimuthal current

$$J^{\hat{\varphi}} = -\frac{R_{gg} \sigma j^{\hat{r}} \sqrt{1 - 2GM/c^2 r} A_{\varphi r}}{r}$$

$$= \frac{R_{gg} \sigma j^{\hat{r}}}{c(1 - 2GM/c^2 r)} \{aM/r^3\}, \quad (43)$$

where

$$A_{r\varphi} = \frac{aM}{cr^2 (1 - 2GM/c^2 r)^{3/2}}. \quad (44)$$

The total current I through $\varphi = const$ plane of plasma is

$$I = \int J^{\hat{\varphi}} r (1 - 2GM/c^2 r)^{-1/2} dr d\theta. \quad (45)$$

One can use approximate behavior for azimuthal current during the rough evaluations:

$$J^{\hat{\varphi}} \approx \frac{R_{gg} \sigma j^{\hat{r}} \omega}{c}. \quad (46)$$

The main question to be answered is how strong the rotational amplification of the electric current can become. For the typical value of parameters $\Omega = 10^3$ s^{-1}, $N = 10^7$ cm^{-3}, $\sigma = 10^7 T^{3/2}$ s^{-1}, $m = 9.1 \times 10^{-28}$ g, $e = 4.8 \times 10^{-10}$ cm$^{1/2}$ g$^{1/2}$ s^{-1} and $c = 3 \times 10^{10}$ cm s^{-1}, the dimensionless parameter $R_{gg} \sigma \omega/c$ can reach big numbers if the temperature of plasma T exceeds, for instance, 10^9–10^{10} K. Such temperatures are realized for pulsar's plasma and in this connection the discussed mechanism of generation of azimuthal current can produce the magnetic field which will compensate the ohmic decay of magnetic field. The temperature dependence is more stronger if the radial current is a consequence of the temperature instabilities, that is if $j^{\hat{r}} = -\sigma \beta \, \text{grad} \, T$ since thermoelectric power β also depends on the temperature as $T^{3/2}$.

Thus the result (42) leads to the deep understanding that even very small radial currents in pulsar's plasma can be essentially amplified by the gravitomagnetic force effects. Furthermore, from our point of view, the observational evidence on the existence of rotational effects on the conduction current provides the laboratory experiment of Vasiliev (1994) where the vertical magnetic field around rotating cylindrical conductor with the radial current has been detected. The experiment has been theoretically explained (Ahmedov 1998) with help of the general relativistic Ohm's law for conduction current which includes rotational and gravitomagnetic terms.

Magnetic field created by the azimuthal current in a current loop in the Schwarzschild and Kerr space times has been considered by Petterson (1974) and Chitre and Vishveshwara (1975), respectively. For a current carrying loop located in the equatorial $\theta = \pi/2$ plane, symmetrically

around the slowly rotating star, at radius b, the magnetic field in the region $r \geq b$ is given by

$$B^{\hat{r}} = -\frac{3\mu \cos\theta}{4M^3}\left[\ln\left(1 - \frac{2M}{r}\right) + \frac{2M}{r}\left(1 + \frac{M}{r}\right)\right],$$

$$B^{\hat{\theta}} = \frac{3\mu \sin\theta}{4M^2 r}\left[\frac{r}{M}\ln\left(1 - \frac{2M}{r}\right) + \left(1 - \frac{2M}{r}\right)^{-1} + 1\right]$$

$$\times \left(1 - \frac{2M}{r}\right)^{-1/2} \tag{47}$$

and identical to the magnetic field of a dipole with moment

$$\mu = \pi b^2 (1 - 2M/b)^{1/2} I. \tag{48}$$

As one can see from (43), (45), (48) the behavior of magnetic field (47) produced by an arbitrary azimuthal current strongly depends on the amplification parameter.

5 Conclusion

We study the equations of motion of two-component plasma embedded in the external gravitational field and derive a generalized Ohm's law for a fully ionized electron-ion plasma within the framework of general relativity. Then combining Maxwell equations and Ohm's law in a stationary and axisymmetric geometry we obtained the approximate (asymptotic) equations that describe the external electromagnetic field and electric current in a plasma (with radial electric current) surrounding a rotating neutron star. By considering a simplified model, we have found that gravitomagnetic effect on radial current may produce increasing azimuthal current in same special conditions in plasma, and therefore a dynamo in principle may be possible. Our general conclusion from the above analysis may be summarized as follows:

1. The general relativistic Ohm's law for plasma contains new terms as compared with special relativity case in the limit of quasi-neutral plasma, which is caused by the gravitomagnetic effects on the electric current.
2. The azimuthal current arises from the gravitomagnetic force effects on radial current in the plasma surrounding rotating neutron star.
3. The most important that due to this gravitomagnetic effect the circulation of azimuthal current is not equal to zero even in a steady state, which in principle may allow axisymmetric dynamo action in some special cases.
4. Rough evaluations for magnetic field arising from the induced azimuthal current for the typical values of parameters give that its quantity may be comparable with the stellar magnetic field only for exceptional cases when the plasma is rare and simultaneously has high temperature.

This result provides us a right to say that Cowling's theorem, in general, can not be violated in the metric of rotating neutron star by the new gravitomagnetic terms in Ohm's law. Thus if the astrophysical object is a rotating compact star, the gravitomagnetic potential and spin of the compact star may, in fact, produce an axial symmetric magnetic field in the surrounding plasma carrying conduction current but the mechanism of its amplification for possible dynamo needs further more detailed investigation.

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